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ABSTRACT

The second volume of this proceedings contains full research articles. Papers include: (1) "Lave and Wenger's social practice theory and teaching and learning school mathematics" (J. Adler); (2) "Being a researcher and being a teacher" (J. Ainley); (3) "Procedural and conceptual aspects of standard algorithms in calculus" (M.B. Ali and D. Tall); (4) "Using small group discussions to gather evidence of mathematical power" (S.E. Anku); (5) "Seventh grade students' algorithmic, intuitive and formal knowledge of multiplication and division of non-negative rational numbers" (A. Barash and R. Klein); (6) "Theory and practice in undergraduate mathematics teaching: A case study" (T. Barnard and C. Morgan); (7) "Heuristics and biases in secondary school students' reasoning about probability" (C. Batanero, L. Serrano, and J.B. Garfield); (8) "Changes in student teacher views of the mathematics teaching/learning process at the secondary school level" (N. Bednarz, L. Gattuso, and C. Mary); (9) "A study of proportional reasoning among seventh grade students" (D. Ben-Chaim, J.T. Fey, and W.M. Fitzgerald); (10) "Children's word meanings and the development of division concepts" (S.B. Berenson and D. Vidakovic); (11) "The use of examples in the teaching and learning of mathematics" (L. Bills); (12) "The influences of significant others on student attitudes to mathematics learning" (A. Bishop, C. Brew, G. Leder, and C. Pearn); (13) "Student teachers' approaches to investigate mathematics: Iterative engagement or disjointed mechanisms?" (P. Blanc and R. Sutherland); (14) "The analgebraic mode of thinking and other errors in word problem solving" (H. Bloedy-Vinner); (15) "Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems" (P. Boero, R. Garuti, E. Lemut, and M.A. Mariotti); (16) "Some dynamic mental processes underlying producing and providing conjectures" (P. Boero, R. Garuti, and M.A. Mariotti); (17) "Teachers' behaviours in teaching with computers" (R.M. Bottino and F. Furinghetti); (18) "A decimal number is a pair of whole numbers" (G. Brekke); (19) "The story of silence: Teacher as researcher--researcher as teacher" (L. Brown and A. Coles); (20) "The cultural evolution of mathematics" (T. Brown); (21) "Beginning learning negative numbers" (A. Bruno and A. Martinon); (22) "Concepts of multiplication of children and teachers in Mozambique" (F. Buikema Draisma); (23) "On preservice teachers' understanding of division with remainder" (S.

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Campbell); (24) "Referent transforming operations: Teachers' solutions" (T. Campos, S. Magina, M.C. Da Cunha, and S. Canoas); (25) "Reconstructing teachers' thinking in teaching problem solving" (O. Chapman); (26) "Bayesian implicative analysis: How to identify the concepts necessary to acquisition of a competence? One example of application to the conceptualization of fraction in teenagers" (C. Charron and J.M. Bernard); (27) "How does addition contribute to the construction of natural numbers in 4- to 13-year-old children?" (C. Charron and N. Ducloy); (28) "Overcoming isolation in teaching" (D. Chazan, D. Ben-Chaim, and J. Gormas); (29) "NESB migrant students studying mathematics: Vietnamese and Italian students in Melbourne" (P. Clarkson); (30) "Levels of formal reasoning in school algebra" (C. Coady and J. Pegg); (31) "Years 2 and 3 children's correct-response mental strategies for addition and subtraction word problems and algorithmic exercises" (T. Cooper, A.M. Heirdsfield, and C.J. Irons); (32) "Why the psychological must consider the social in promoting equity and social justice in mathematics education" (T. Cotton and P. Gates); (33) "Clinical analysis of difficulties in algebraic problem solving among Brazilian students: Principal aspects and didactic issues" (J.-T. Da Rocha Falcao); (34) "What is the difference between remembering someone posting a letter and remembering the square root of 2" (G.E. Davis); (35) "The illusion of linearity: A persistent obstacle in pupils' thinking about problems involving length and area of similar plane figures" (D. De Bock, L. Verschaffel, and D. Janssens); (36) "The development of whole-part relationship in situations of combine structure for a student with learning disabilities" (L. Deblois); (37) "Study of the evolution of graduate students' concept images while learning the notions of limit and continuity" (C. Delgado and C. Azcarate); (38) "Facets and layers of the function concept" (P. DeMarois and D. Tall); (39) "Evolution of conceptions in a mathematical modelling educational context" (G. Dettori, R. Garuti, E. Lemut, and M.A. Mariotti); (40) "The role of physical phenomena in students' investigations of the tangent function" (H.M. Doerr); (41) "Written subtraction in Mozambican schools" (J. Draisma); (42) "Children's reasoning in solving novel problems of deduction" (L. English); (43) "Association judgments in the comparison of two samples" (A. Estepa and F.T. Sanchez); (44) "On some factors affecting advanced algebraic problem solving" (P.L. Ferrari); (45) "Intuitions and schemata in probabilistic thinking" (E. Fischbein and D. Schnarch); (46) "The Fennema-Sherman 'mathematics as a male domain scale': A re-examination" (H.J. Forgasz, G.C. Leder, and P.L. Gardner); (47) "Students' misconceptions with decimal numbers. Preliminary results from a study of computer based teaching" (A.B. Fuglestad); (48) "Qualitative analysis in the study of negative numbers" (A. Gallardo); (49) "An analysis of some students' concept images related to the linear function when using a 'generic organizer'" (M.C. Garcia and C. Azcarate); (50) "Learning proportion using a computer environment in the classroom" (M. Garcia Mila, E. Marti, C. Gomez-Granell, and B. Steren); (51) "Development of concepts of the arithmetic average from high school to university" (L. Gattuso and C. Mary);(52) "Selling mathematics: The language of persuasion in an introductory calculus course" (S. Gerofsky); and (53) "Mathematical concepts, their meanings and understanding" (J.D. Godino). (ASK)

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Volume 2

International Group for the Psychology of Mathematics Education

PME 20

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Edited by

Luis Puig
Angel Gutiérrez

Volume 2

Valencia (Spain), July 8 - 12, 1996

Universitat de València
Dept. de Didàctica de la Matemàtica

Proceedings of the 20th Conference of the International
Group for the Psychology of Mathematics Education
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RESEARCH REPORTS

LAVE AND WENGER'S SOCIAL PRACTICE THEORY AND TEACHING AND LEARNING SCHOOL MATHEMATICS

Jill Adler, University of the Witwatersrand

In this paper I argue that Lave and Wenger's social practice theory offers a very powerful language for understanding knowing and learning about and the practice of teaching. However, this theory does not transfer unproblematically into knowing and learning about the practice of school mathematics. This argument arises within a study on teachers' knowledge of their practices in multilingual mathematics classrooms, a study that requires theorising knowledgeablility of school mathematics teaching, that is, of both 'teaching' and 'school mathematics'. The implications of this argument for research is that Lave and Wenger's social practice theory needs elaboration if it is to successfully illuminate learning and knowing school mathematics.

INTRODUCTION

Lave and Wenger's theory of social practice (1991) has recently gained currency in PME. It has been invoked to examine, describe and explain mathematics learning in school (See, for example, Jaworsky, 1994; Meira, 1995). In this paper I will elaborate Lave and Wenger's theory and argue why their notion of learning through participation in communities of practice appropriately and powerfully illuminates learning and knowledge about teaching. But a shift into school learning raises questions about what constitutes a community of practice, and hence about theorising the learning and knowing of mathematics in school within social practice theory.

SITUATING LEARNING IN PARTICIPATION IN COMMUNITIES OF SOCIAL PRACTICE

Lave (1991) and Lave and Wenger (1991) situate learning in communities of social practice. Building on Lave's earlier work on situated cognition (1985; 1988), they develop a theory of social practice - what they call 'legitimate peripheral participation in communities of practice' (LPP). LPP can illuminate how teachers learn about teaching, their knowledge about teaching and provides a theoretical orientation to teachers' knowledge that incorporates the personal, the practical and the social.

> Briefly, a theory of social practice emphasizes the relational interdependency of agent and world, activity, meaning, cognition, learning and knowing. It emphasises the inherently socially negotiated character of meaning and the interested, concerned character of the thought and actions of persons-in-activity...In a theory of practice, cognition and communication in, and with, the social world are situated in the historical development of ongoing activity. (pp.50-51)

For Lave and Wenger, becoming knowledgeable is a simultaneous and ongoing fashioning of personal and professional identity within a community of social practice. Learning is located in the process of co-participation, and not in the heads of individuals. This is thus a social theory of mind where meaning production is taken out of the heads of individual speakers and located in social arenas that are at once situationally specific and in the broader society. In Lave and Wenger's

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terms, knowledge about teaching is thus fundamentally tied to the context of teaching, and cannot be abstracted out. Knowledge about teaching is also dynamic and simultaneously personal and social.

'Legitimate peripheral participation' (LPP) is the conceptual bridge between the person and the community of practice. As people participate in communities of practice so they become more knowledgeable in the practice, they move from a position of **'newcomers'** to becoming **'old-timers'** with greater mastery of the practice and with all the conflicts, contradictions, changes and stability that entails. LPP is a means of explaining both the developing identity of persons in the world, and the production and reproduction of the community of practice. Here is a conceptual framework for integrating the personal and the social in describing and explaining teaching.

For Lave and Wenger, social practice, and not learning, is their starting point. Learning is rather a dimension of any social practice. It is at once subjective and objective through a focus on whole person-in-the-world. Learning is increasing participation in communities of practices and concerns the whole person acting in the world. This is in sharp contrast to dominant learning theory which is concerned with internalisation of knowledge forms and their transfer to and application in a range of contexts. Knowing is thus an activity by specific people in specific circumstances. Identity, knowing and social membership entail one another. Thus 'learning is not a condition for memberships, but is itself an evolving form of membership' (p.53). Knowing about teaching and becoming a teacher evolve, and are deeply interwoven in ongoing activity in the practice of teaching. Knowledge about teaching is not acquired in courses about teaching, but in ongoing participation in the teaching community in which such courses might be a part.

This view of knowledgeability opens another way of understanding teachers' roles in developing knowledge about teaching. Debates on the 'teacher-as-researcher' often polarise researchers and teacher-researchers, with arguments about what constitutes research, and, moreover, what knowledge about teaching in fact affects practice[1]. Lave and Wenger's social practice theory clearly identifies teachers as a crucial source of knowledge about teaching.[2]

Lave and Wenger distinguish between **peripheral** and **full** participation where both are legitimate but different forms of participation in the practice and both are constantly changing. Full participation signals mastery in the form of full membership in the practice rather than an endpoint in learning/knowing all there is to know about the practice. The process of moving from peripheral to full participation thus requires a 'decentering' of mastery and pedagogy away from the individual master or learner and into the structuring of resources in the community of practice (p.94). Learning and mastery are a function of how resources are made available. For Lave and Wenger understanding participation and learning requires a focus on the learning curriculum, and not the teaching curriculum. It is neither teaching intentions, nor planned pedagogy that can both enable and explain learning. Rather, the social structure of the practice and conditions for legitimacy define the practice and possibilities for learning.

Peripheral and full participation provide a means for distinguishing new and older teachers, as well as for distinguishing within newer or older teachers in such a way that those that remain more peripheral teachers are not so simply because they are 'poor'. This might well be the case, but must be seen in relation to a teacher's access to resources in the social structure of teaching. The concept of transparency elaborates this point.

TRANSPARENCY

For Lave and Wenger, becoming more knowledgeable, entails having access to a wide range of ongoing activity in the practice - access to old-timers, other members, to information, resources and opportunities for participation. Such access hinges on the concept of **transparency** (p.100).

> 'The significance of artifacts in the full complexity of their relations with the practice can be more or less *transparent* to learners. Transparency in its simplest form may imply that the inner workings of an artifact are available for the learner's inspection...transparency refers to the way in which using artifacts and understanding their significance interact to become one learning process. (p.102)

Becoming a full participant means engaging with the technologies of everyday practices in the community, as well as participating in its social relations. Thus, access to and use of artifacts in the community is crucial. Often material tools, artifacts -technologies - are treated as given. Yet, often they embody inner workings tied with the history and development of the practice - these need to be made available.

Lave and Wenger elaborate 'transparency' as involving the dual characteristics of **invisibility and visibility**:

> ... invisibility in the form of unproblematic interpretation and integration (of the artifact) into activity, and visibility in the form of extended access to information. This is not a simple dichotomous distinction, since these two crucial characteristics are in a complex interplay. (p.102)

In other words, the invisibility of mediating technologies is necessary for focus on and supporting the visibility of the subject matter. The notion of transparency connects with the implicit and explicit in pedagogical relations. The implicit can enable a focus of attention on the subject matter. But for effect, it must make the subject matter visible. Often, again for cultural reasons, implicit pedagogical rules can obstruct rather than enable the visibility of subject matter. It is the implicit rules that become the object of attention, rather than the subject matter.

In short, practices that are more or less transparent can enable or deny access to the practice - enable/legitimate or obstruct/prevent peripheral participation. Through transparency, members can exercise control and selection into the practice. Thus, the explanatory burden for learning - and here learning about teaching - is placed in cultural practice, in the community of teaching, and not on one kind of learning or another. Increasing participation and hence knowledgeability is not about connecting theory and practice, or experience and abstraction, but rather entails the organisation of activities in teaching that makes their meaning visible.

LEARNING *TO* TALK

In addition to transparency, legitimate peripheral participation also involves learning how to talk (and be silent) in the manner of full participants. For newcomers then, the purpose is not to learn from talk as a substitute for legitimate peripheral participation, it is to learn to talk as a key to LPP. Unpacking these concepts related to talk, Lave and Wenger distinguish between **talking within and talking about** a practice. Full participation in a community of practice means learning *to* talk, and this entails talking about and within the practice (p 109). Talking about the practice from the outside is what often constitutes formal learning (eg. theory of education in teacher education) where student teachers learn to talk about teaching from outside the practice. For Lave and Wenger this is achieved through a didactic use of language, not itself the discourse of teaching practice, and thus creates a new linguistic practice all of its own.

Talking within and talking about practice thus need redefinition (p.109). Talking within a practice itself includes both talking within (eg exchanging information necessary to the progress of ongoing activities) and talking about (eg stories, community lore). Inside the shared practice, both forms of talk fulfil specific functions: engaging, focusing and shifting attentions, bringing about co-ordination on the one hand; supporting communal forms of memory and reflection as well as signalling memberships on the other.

Talking about a practice also usually involves both talking within and about - but in Lave and Wenger's terms, the effect of this talk is not full membership of the practice - because it is happening from the outside - it is rather what they call 'sequestration' and an alienation from, or prevention of access to, the practice.

We know only too well from teacher education courses that a prospective teacher's ability to write a good essay on what is good teaching - where 'good essay' is signalled in the practices of the academy - often bears little relation to good teaching in practice.

Knowledge about teaching is thus not simply in individual teachers' heads: it is tied to their identities and evolves in and through co-participation in the practices of the teaching community. Teachers, particularly if they have been in practice for some time, are more or less knowledgeable about their practice (teaching) depending on the community, their access to its resources - particularly to activities related to talking within and about the practice, and to the transparency in the practice.

It is this conception of teacher knowledgeablility that that has shaped my own study of teachers' knowledge of their practices in multilingual mathematics classrooms. Teachers have knowledge to share about teaching mathematics in multilingual mathematics classrooms. Moreover, a study that wishes to access such knowledge should then include teachers talking about *and* within their practices. In short, Lave and Wenger provide a theoretical orientation, with design implications, for a study entailing teachers' knowledge.

However, a study of teachers' knowledge of the teaching and learning of mathematics in school needs also to theorise the learning and knowing of school mathematics. Does Lave and Wenger's social practice theory transfer from apprenticeships and other communities of practice like Alcoholics Anonymous and teaching into school mathematics learning?

SHIFTING INTO SCHOOL LEARNING

Lave and Wenger develop their understanding of learning as part of social practice through contexts of successful learning - apprenticeships. They explicitly turn away from the school because learning as intended in schools has been not only been unsuccessful for so many, lack of success has also been socially distributed. In addition, the formal school has been the dominant and determining domain of learning theory, yet it is not the only context of learning.

Instead of teachers and learners we have **old-timers** - knowledgeable others in a community of practice - and **new-comers** whose knowledge and identity evolve through centripetal participation in the practice. They elaborate the importance of transparency in the practice and access to resources for newcomers becoming knowledgeable and fashioning a successful identity. I have argued that this conceptualisation of learning within social practice assists the theorising of knowledge about teaching - how teachers learn about teaching. How does Lave and Wenger's conceptualising transfer to theorising learning mathematics (for example) in school? In Lave and Wenger's own terms this question is important: school is a specific social context, involving different social practices to contexts of apprenticeship.

A shift into school learning, raises a number of questions: What/who is the community of practice in school mathematics? What is the community that teachers are old-timers in? mathematicians? mathematics teachers? Or are older students, or mathematically schooled adults the old-timers here? and where are they in relation to the teachers? and pupils? What are pupils newcomers into? What might constitute legitimate peripheral participation in the mathematics classroom and towards what is the centripetal process of participation? becoming a mathematician? a mathematically schooled adult[3]?

Lave and Wenger offer a general theory of social practice in which learning is always a part. However, there are clear difficulties moving into the context of schooling. In school, students remain students until they leave. No matter how much mastery they might have achieved, only a few, after school, might become their mathematics teachers and even fewer mathematicians. Moreover, their teachers - however mathematical - are not, in the context of schooling, practising mathematicians. There is also a labour intensity in an apprenticeship model that does not transfer easily to mass schooling conditions. Thus, while Lave and Wenger's intentions are for a general theorising, and they attend, at moments, to the specificity of schooling (pp,39-41), they, in fact, sidestep difficulties in using their conceptualisation to interpret and explain teaching and learning in school.

Difficulties in interpretation can be located in their privileging the structure of the practice rather than the structure of pedagogy as the source of learning. Motivation, identity, conflict, power relation all reside in the community of practice and will work in different ways to enable centripetal movement to full participation or constrain it. This is why for them, learning is only understood in relation to a learning rather than a teaching curriculum. But in so doing, and despite their own commitment to move away from dichotomies, they insert a new and equally problematic dichotomy between teaching and learning.

It is useful to ponder for a moment that in Russian for example there is only one word - obuchenie - that describes teaching/learning. In other words there is no learning without teaching and vice versa. The teaching/learning relation is a hugely complex one. It is as fundamental a problem in teacher education as it is in school learning. Dominant teacher education practices are structured in both the academy and in the school itself - a combination of a formal and an apprenticeship context. The success of this combination and the relative merits, weightings, contents and processes of the two parts remain the focus of ongoing research and debate. Lave and Wenger's theory of social practice shifts the problematic away from theory/practice dichotomies and questions of transfer and encourages us rather to examine the resources made available in different contexts of teacher education and their possible effects.

I have argued that while Lave and Wenger provide a framework for understanding teachers knowledge and identity, their social practice theory is not unproblematically transferable to school learning and teaching. They have, nevertheless, constructed useful concepts that could provoke interesting insights into learning and teaching mathematics in school. Specifically, **access and sequestration**, the availability of **learning resources**, **transparency**, and their distinction between **talking within and about** a practice are easily read into the pedagogical relation in maths teaching in school, and are thus useful to explore further.

In relation to transparency, and, for example, in a study on mathematics learning in multilingual settings, language - and specifically speech - functions as a tool in the classroom. A great deal of classroom communication occurs through speech. Speech is thus a resource where, in Lave and Wenger's terms, invisibility and visibility are in constant interplay: speech should be invisible so that the subject of inquiry - a mathematical problem, say - can be engaged i.e. become visible. But language is a cultural tool and never unproblematic. In and of itself, it can mediate the activity in the course of action. For example, a group of learners working on a problem communicate through verbal speech, gestures and so on. This communication is supposed to make the problem more visible, more accessible. But the social relations in the discussion and the discussion itself can become the object and focus of attention, particularly if it occurs in a mix of languages. That language itself can become visible, mediate activity so as to obscure the task rather than make it visible seems fairly obvious in a multilingual class. Thus, what Lave and Wenger powerfully illuminate is that resources for learning, like language, can enable or exclude. Depending on how they are used, resources can enable access to the practice or sequester participants.

For Lave and Wenger, becoming knowledgeable in a practice entails learning to talk within and about the practice, and not learning from talk. Yet a great deal of literature of language and learning is about both - learning to talk and learning from talk. But Lave and Wenger's distinction between talking within and about is useful. First, it links with distinctions between talk as exploratory/expressive vs talk for exhibiting/displaying knowledge. What might this mean in a maths classroom? In classrooms where there is a move to more exploratory problem-solving mathematical practices, students often work together on tasks, and then report on their working to others in the class and to the teachers. While on tasks, pupils could be said to have opportunity for talking **within** their mathematical practice.

Then, and either to the teacher, or other pupils or both, they talk **about** their mathematical ideas. Thus they are being provided opportunity to learn *to* talk but a question begging is: given the distinct practice that is school mathematics - that classroom talk has its own form and function (Mercer, 1995), how then are pupils apprenticed into this talking? And what happens in classes where children have a range of spoken languages? In short, what Lave and Wenger's theorising of learning does not explain, is the specific demands of apprenticeship into school mathematics, and its necessary focus on the structure of pedagogy.

Within a social theory of mind, that is sharing some basic assumptions with Lave and Wenger, there has been a great deal of research, theorising and debate on the mediation of mathematical knowledge in school. It is way beyond the scope of this paper to elaborate fully here. Briefly, however, more sociological arguments draw on the work of Paul Dowling (See, for example, Davis and Coombe, 1995) and the importance of the discursive elaboration of mathematical knowledge in the classroom for access or apprenticeship into mathematics as opposed to widespread alienation. Here, mediation of mathematical knowledge via the everyday and the emphasis on procedural knowledge in the curriculum come under scrutiny. More psychologically oriented research has focused on the question of meaning where both children's meanings and socially constructed mathematical knowledge are important in the pedagogical situation. Alienation is a function of the suppressing or ignoring of learner meanings. Informed by both neo-Piagetian and socio-cultural theory, here, quality and effective mathematic learning and teaching in school involves a blending of both self- and other-mediated activity, between scaffolding a task and providing for creative responses to a task, between teaching and learning (see, for example, Cobb, 1994; Confrey, 1994, 1995a, 1995b).

CONCLUSION

In short, explaining access to or sequestration/alienation from school mathematics requires an understanding of the structure of pedagogy. Lave and Wenger's social practice theory falls short here. My own study of teachers' knowledge of their practices in multilingual mathematics classrooms combines social practice theory with socio-cultural theory for a full and effective elaboration of knowing, learning and teaching mathematics in school.

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NOTES

1. See, for example, Crawford and Adler, 1996; Cochrane-Smith and Lytle, 1993 and Richardson, 1994.

2. I am not suggesting that only teachers can know about teaching. Rather, we can and must learn about teaching from teachers themselves.

3. Dowling's (1990) analysis of mathematics in the distinct domains of the everyday, the school and the academy is useful here.

BEING A RESEARCHER AND BEING A TEACHER

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This paper seeks to explore the tensions between the roles of teacher and research, and between different styles of research activity, through the detailed study of accounts of a particular classroom incident. Personal observations and reflections are used, following the style of the Discipline of Noticing (Mason, 1994), to consider what may be learned from these tensions to inform the activities of both teaching and researching.

Introduction

The subject of this paper arises from personal experiences of my involvement in a long term school-based research project. I am concerned to try to explore the tensions between the role of teacher and the role of researcher in ways which may shed light on my effectiveness in both roles. The literature on educational research abounds in texts on classroom observation, and on the teacher as researcher (Hitchcock and Hughes (1995), Hopkins (1993) and Hammersley (1986) are typical examples). What is less easy to find is any literature which deals with the researcher as teacher, or with the boundaries between the two roles in the classroom. My interest here is not in the macro level of research design, but in the micro level of individual interactions in the classroom. The activity I am engaged in is very much 'researching from the inside' employing the Discipline of Noticing (Mason (1994)).

The research context

The research project I am working on is concerned with exploring the effects of high levels of access to portable computers on children's mathematical learning. It is based in a primary school, involving children aged 6 to 11 years. Our research takes place mainly in normal classroom settings with a whole class of children. The activities used arise as far as possible from the normal work planned for the class, but with input from the researchers (myself and Dave Pratt) to extend and enrich the mathematical ideas involved through the use of appropriate software.

In the early stages of the project, we worked with three classteachers[1] over a period of two years. It was an important part of the project design that this period would be one of considerable professional development for the classteachers, in terms of both their confidence with computers, and their knowledge of mathematics, in which none of them were specialists. We also planned activities with them in school, and visited their classrooms regularly to observe and collect data.

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[1] The term 'classteacher' is used throughout to denote the regular members of the school staff involved in the project. This is distinguished from 'teacher', which refers to the person taking the role of teacher at a particular time.

During this time we developed good professional relationships, and became familiar with each other's approaches to teaching. Without the friendship which built up between the project team during this period, much of the classroom-based research which we undertook would have been very difficult to carry out, particularly as the classteachers were working in areas which were largely new to them. It was as important that we had confidence in each other as teachers, as that the classteachers trusted us as researchers. Although our main interest is in cognitive issues, studying how children's mathematical learning is affected by the use of the computer, our data was collected largely through a more ethnographic approach, observing lessons, collecting examples of children's work, interviewing teachers and children.

During a later, more intensive, phase of the project, we were able to work full time on the project for one year. Each of us attached to one class during a school term, teaching mathematics and science for 3 half days per week. The timetable was arranged so that the other lecturer was able to act as a researcher during these lessons. The classteachers were generally present during the lessons, sometimes working alongside the 'teacher', and sometimes acting as a second 'researcher'. Thus the roles of teacher and researcher were clearly defined for us within any particular lesson, but the transitions between roles was frequent. It is this experience of acting both as a teacher and as a researcher which has focused my attention on tensions involved in the relationship between the two roles.

Conflicting perceptions of a classroom incident

I shall work mainly on a single incident which had a significant effect on my awareness of the tensions arising from perceptions of classroom roles. Data has been drawn partly from aspects of the data collected as part of the main project: from field notes taken by researchers, and the journals kept by lecturers and classteachers. The incident took place at an early stage in the project, before the intensive period of work described above, and involved a classteacher, whom I shall call Martha, and both lecturers.

Martha was using laptop computers with her class for the first time, working on a data handling activity which had been planned jointly with other members of the project team. The children had gained reasonable confidence with the machines, and Martha had introduced the question *'What affects how a toy car rolls down a slope?'* We have reported elsewhere on some of the children's work on this activity, and on the classteacher's own insights about the approach to collecting and handling data

(Ainley and Pratt (1994, 1995)). The following sections give three different views of activity in the classroom during a particular week.

Janet's story

Visiting the classroom, I tried to take a back-seat, allowing Martha to take the lead in the lesson. I sat with groups of children, taking field notes on my own laptop. Extracts from my notes reflect my attempts to record my observations without getting drawn into 'teaching' or solving technical problems.

> *I'm sitting at a table, but because I'm writing on the machine and looking at the screen I feel more invisible than I would with pencil and paper. ... It's very hard not to get drawn into problem solving, so I have come away to three pairs of girls working in the book corner. ... The girls with the extended ramp I watched on Tuesday are busy testing, with a fine disregard for accuracy. They are holding the ramp in place, but not noticing that they keep moving it. They have so much information to collect about each car that it is a very long process. ... Some of the ramps are so steep that they cannot record information easily.* (Janet's notes 930114)

Both the style of the activity and the use of the technology were new to Martha and to the children, and so it was understandable, and for me expected, that there would be some time spent exploring less profitable approaches before much progress was made. However, as time went on my notes reflect some anxiety about how the activity was developing.

> *About 2.15. Some* [groups]*are still typing in their field names. Most have 4 or 5 records at most. Martha is stressing that they need lots of data, but in practice this is going to take a long time. I'm not sure how much data handling is going to happen.* (Janet's notes 930114)

Re-reading my notes evokes a strong sense of the discomfort I felt at this point. As a researcher, I felt frustrated that time was passing and nothing very interesting (mathematically) was happening. I was aware that I had other calls on my time which might prevent me seeing later stages of the activity.

At the same time the part of me that is a teacher felt that I would have done things differently, that the lesson was losing momentum and that the children needed some clear direction in order to move on. However, I was very aware that it was not my classroom: I was *not* the teacher, and had a strong sense of an etiquette which did not allow me to intervene. I was not sure how clear Martha's mathematical and scientific understanding of the situation was. At this point in our relationship I often found it hard to read her reactions, as a previous journal entry indicated.

> *My heart sank a bit at Martha's introduction, which was very brief. ... I would have wanted to let them play with the cars first, and then spend more time discussing possible variables, ... However, as things progressed I revised my opinion. ... I must be careful not to underestimate Martha!* (Janet's journal 930112).

In the classroom, my tensions were soon resolved.

> *As usual, just as I was wondering if Martha has realised things are going a bit awry, she came and talked about it. ... Martha feels a bit at sea, I'm going to try talking to the whole group ...* (Janet's notes 930114).

Stepping into the role of teacher made me feel much better. I was able to take control of the direction of the children's work and pull together ideas for their future investigations, even though I was not going to be there the following day to see the results. In a fairly short discussion I felt that they had made some progress. Although I had had some worries about how things were progressing in Martha's classroom, I felt at this point that the situation had been partially retrieved. I discussed the situation with Dave, my colleague, who would be making the next visit to Martha's classroom.

Dave's story

Extracts from Dave's journal and field notes indicate how the next lesson progressed, but also give some insights into a different perspective on the roles of researcher and teacher. Dave is prepared to intervene more explicitly to influence Martha's planning, though like me, he finds that he has possibly underestimated her perceptiveness about the situation.

> *Yesterday Janet worried me by her report of the previous day. It seems that Martha had found problems helping the children through the scientific process involved in the experiments ... Janet and I decided that I would get in early and try to talk to Martha about how the children might be focused more and that this could result in them using the spreadsheet instead* [of the database]*which they would find easier. In practice, Martha herself had come to much the same conclusions.* (Dave's journal 930115)

As the lesson progressed, with children now working with spreadsheets to investigate the effects of just one variable, Dave seems to see his role as being a teacher as much as being a researcher. His field notes are all written in the past tense, describing incidents that have taken place, and in which he has been involved, rather than trying to record events as they happen. Dave intervenes directly to show children techniques on the computer, and also takes the initiative in suggesting to Martha how the activity might proceed.

> *Larger groups were formed by merging all those who wanted to do the same thing. There were two weight groups, one surface group, and one ramp height group. ... I had talked to Martha about the need to keep all other things the same. In fact then they might as well use spreadsheets. Martha was unsure about how to use the spreadsheet. However, after I had shown one group, she was clearly much happier and was able to see how to teach the other groups.* (Dave's notes 930115)

Despite the fact that Dave and I regularly looked through and discussed each other's field notes, these differences in research style did not become apparent to us. Our attention was generally focused on what the children had done, and issues to do with the mathematics or the technology. It was only at a later stage, when we were both acting in the roles of both teacher and researcher, that I began to reflect on and then explicitly discuss the tensions raised. However, Martha herself was much more perceptive about our approaches.

Martha's story

For Martha, this incident was something of a turning point, at which she might easily have rejected the project because of the pressures it was putting upon her. She reflected her conflicting feelings openly in her journal, despite knowing that we would eventually read it.

> *I'm not sure what to say about today. At the moment I feel clearer again about the situation but at 2.30 this afternoon I felt confused and very dissatisfied about the whole thing, and wishing I had never heard of lap-tops.* (Martha's journal 930114)

With typically disarming honesty, she also commented on our behaviour.

> *I do feel that this phase of the project is being approached differently by Dave and Janet. Janet is really sitting back and taking on the role of the observer rather than supporter, helping only when I am desperate. I wonder why?* (Martha's journal 930114)

> *I felt much clearer this morning about the task for today and the way ahead. ... I talked things through with Dave before we started and he seemed to think my plan was workable, and that we, both the children and myself, needed to go through that rather busy and confused stage. I realised that I definitely had learned a great deal. ... I wonder if Janet allowed me to go down the wrong path intentionally? ... I felt much more comfortable with Dave taking a more active role with the children. With Janet, when she is just observing, I feel as if I am as much the guinea pig as the children and the computers are- (I know I am really) but it feels as if I'm on teaching practice.* (Martha's journal 930115)

Reflections

At first, I found reading Martha's journal hurtful. It presented an image of myself which I did not recognise, and which I felt was unfair. Now that we know each other much better, Martha and I have been able to discuss this incident several times with good humour: it has become known as 'that Thursday afternoon'.

Returning to her written comments now, I am struck again by their perceptiveness, heightened perhaps by contrast to my lack of it. As a researcher, I made a deliberate attempt *not* to intervene or take any part in the teaching or organisation of the lesson. I had, wrongly, assumed that she might feel threatened if I behaved like another teacher in her classroom, ignoring other connotations of my behaviour.

Martha also comments on differences in the ways in which Dave and I acted in the role of researcher in the classroom. These differences became more apparent to me when I was being a teacher, with researchers in my classroom. In simplistic terms I would characterise two research approaches, reflected in Martha's comments, as those of *observer* and *experimenter*, illustrated briefly in the table below.

Observer	**Experimenter**
• passive - monitoring activities, but not intervening, using the teacher as an agent	• active - intervening to make an input to the activity, to see what happens
• trying to record everything, without too much filtering	• focusing on recording what is most interesting
• holding back - not wanting to invade the teacher's territory	• getting involved - fitting into the territory by behaving like a teacher
• minimising the effect of the researcher	• deliberately acting as a catalyst

These are not intended as clear-cut categories. Certainly neither Dave or I feel our behaviours fitted entirely into one column or the other, but the polarisation serves to expose the often subtle distinctions more clearly. The two styles are also seen more clearly in the context of the reactions of teachers to the presence of the researcher.

We saw earlier that Martha felt more comfortable with an *experimenter* in her classroom than with an *observer*. One story I can tell for this, in retrospect, is that an observer reminded her of being assessed (even though the observer's attention was on the children), while an experimenter felt like having another teacher working alongside her - a familiar situation which had positive associations.

As a teacher, I sometimes felt resentful of an *experimenter* in my classroom: I felt that my control of the overall direction of the lesson was being undermined. In discussion, Dave (as a teacher) has reported times of frustration at the presence of an *observer*, feeling that without active intervention on the part of the researcher to move children's thinking on, opportunities were being missed.

As a researcher, my reactions to the styles of observer and experimenter are less clear-cut. I often feel uncomfortable as an *observer*. To watch children's activity and not join in feels false, unnatural. I have a sense that I am not doing anything. (This feeling has resonances in the experience of standing back as a teacher to assess what is happening in the classroom. There may be echoes here of Martha's reaction - why isn't she helping?) However, at another level, I know that what I am doing is important. The significance of children's words and actions are not always immediately apparent: it is only through detailed and uncritical observation that they can be captured. Mason (1994) stresses the importance of 'giving an account of' before 'accounting for'. As an observer I have this model in my head. I try - often unsuccessfully - to record incidents without judgement.

There is an attempt here to eliminate the researcher from the research context: to create an invisible, neutral monitor, keeping the subject of the research 'clean'. This image is appealing; clinical, efficient, 'correct'. There is a sense in which we would perhaps all like our research to be seen in this light. However, in our project the researchers were not neutral observers, but active participants in shaping the research context. Whatever we did or did not do in the classroom, we had been involved in planning the activities that the children worked on, often giving quite explicit models to the teacher as to how a new stage of the activity should be introduced. In this sense, assuming the role of *observer* in the classroom was to some extent a pretence.

In contrast, I often find acting as a *experimenter* when I am in the classroom more comfortable: I feel I am doing something, and getting some responses to my actions. There is a great satisfaction in making a comment or asking a question, and recording the effects on children's activity. I find the role of experimenter seductive: I use the word deliberately to convey both the pleasure and the lingering sense of unease. It is this unease which I want to explore further.

The rationale which Dave and I have discussed for acting as an experimenter is that we have already set up the learning situation and want to see its effects. Specific interventions in the classroom are made as a result of observing children's progress and judging that they are in need of further input to challenge or extend their understanding. Having made the intervention, the researcher can then withdraw to observe the effects of the intervention. The tension for me lies in the discipline required to make this withdrawal. Once I begin to intervene, I find myself becoming a teacher.

Having begun by looking at the interactions between researchers and teachers, I end by looking at the relationship between the researcher and the teacher *in myself*. My mental image is of stepping across a line between two areas of activity. Sometimes the step is deliberate; sometimes inattentive wandering. Once I step over that line and begin to be a teacher, I find it hard to act effectively as a researcher. I have an investment in the children's success, and I am looking for evidence of this. I stop seeing and hearing what they really do so clearly. Children generally don't feel able to say when they have had enough of my intervention: indeed they may be happy for me to carry them along my line of thinking. The purpose of my intervention as a researcher will probably be different. I want to have an effect, but to do as little as possible, leaving space to listen to the children. I may ask for the children's assent to my intervention, and try to leave them the freedom to ignore it.

Far from leading me to feel that I must deny my identity as a teacher in order to be an effective researcher, I see the skills that I have as a teacher as crucial in enabling me to frame such interventions effectively. At many levels, I can not stop being a teacher when I am in school. To be an effective researcher (and perhaps also an effective teacher) I believe that I need to be aware of the attractions and constraints of both roles.

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Procedural and Conceptual Aspects of Standard Algorithms in Calculus

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This research studies the different methods students use to carry out algorithms for differentiation and integration. Following Krutetskii, it might be conjectured that the higher attainers produce curtailed solutions giving the answer in a smaller number of steps. However, in the population studied (Malaysian students in the 50th to 90th percentile), some higher attaining students wrote out solutions in great detail, so little correlation was found between the attainment of students and the number of steps taken. On the other hand, the higher attainers had less fragile knowledge structures and were significantly more likely to succeed. But with problems that can be simplified by a non-algorithmic manipulation before using a standard algorithm, the higher attainers were more likely to use some form of conceptual preparation.

Introduction

In his renowned study of the different problem-solving styles of children, Krutetskii (1976) showed that, of four groups (gifted, capable, average, incapable), the gifted were likely to *curtail* solutions to solve them in a small number of powerful steps, whilst the capable and average were more likely to learn to curtail solutions only after considerable practice, and the incapable were likely to fail. This may be related to the strength of the conceptual links formed by the more successful students in their cognitive structure (Hiebert and Lefevre, 1986) which helps the individual utilise knowledge in an efficient and powerful way.

The brain is a huge simultaneous processing system that must filter out most of its activity to be able to focus attention on a small amount of data for decision making (Crick, 1994, p. 61). Therefore the ability to code information efficiently—to make appropriate links between concepts and to develop methods that economise on processes—is likely to increase the brain's capacity to perform mathematics.

Davis (1983) suggested that at least two kinds of procedures exist: a visually moderated sequence (VMS) and an integrated sequence. In a VMS, the whole sequence is not yet apparent and the student carries out a manipulation to produce new written information which is then operated on in turn until the problem is solved. In an integrated sequence, the student is aware of the whole algorithm built up from smaller component sequences.

Hiebert and Lefevre *et al* (1986) contrasted *procedural* and *conceptual* methods of processing mathematical information. Following Dubinsky (1991) and Sfard (1991), who focused on the way in which process becomes encapsulated (or reified) as mental object, Gray & Tall (1991, 1994) introduced the notion of *procept*: the amalgam of *process* and *concept* represented by the same symbol. They hypothesised that less flexible thinkers see the symbol more as a process to be carried out using fairly inflexible procedures. The more flexible thinkers are hypothesised to view a symbol both as a process to *do* mathematics and a concept to *think* about. Evidence with young children doing arithmetic showed that whilst the less successful clung to (often idiosyncratic and inefficient) counting procedures, the more successful not only showed flexible ways of thinking conceptually, but also often chose more efficient procedures to carry out required processes.

In this study we consider a population of students solving problems involving standard algorithms in differentiation and integration. Three groups, each of twelve students, were selected attaining grades A, B, C respectively in recent examinations. Following Krutetskii, one might hypothesise that the more successful make sophisticated links to reduce the manipulation involved and curtail their algorithms to make them more effective, whilst the less successful are likely to use more rigid procedural methods that have longer and more fragile connections which may break down. However, the population studied does not fully reflect these hypotheses. It consists of Malaysian students following degrees involving mathematics taken from the 50th to the 90th percentile of the total population (because the highest attaining 10% travel abroad to study). It was found that in this population there was little correlation between attainment and curtailment of solutions (because the higher attainers included those who wrote out painstakingly detailed solutions). The major difference between higher and lower attainers in standard questions was that the low attainers had more fragile connections in their knowledge structure and were more likely to break down.

However, the higher attaining grade A students were more likely to show the capacity to use subtle initial simplifications to simplify the overall manipulation required. Specially designed problems, such as finding the derivative of $\frac{1+x^2}{x^2}$ benefit from an initial *conceptual preparation* to make the differentiation algorithm simpler to apply. Those who fail to carry out a conceptual preparation and tackle the problem using the standard algorithm may not only be applying a more complex algorithm, but have to follow it up with a more complex *post-algorithmic simplification*.

It was found that in certain questions, higher attainers were more likely to use conceptual preparation than lower attainers. On other occasions where the preparation required was more subtle or the gain was not so obvious, their confidence in symbolic manipulation led some high attainers to use a standard method even when they were aware of a possible alternative. Just as with the more successful children in arithmetic, who would confidently use efficient procedures when they did not immediately recall the relevant number facts, the more successful calculus students developed a powerful combination of conceptual and procedural methods whilst the less successful were often faced with a more difficult manipulation and therefore were more likely to fail.

Curtailment of solutions

A crude method of determining the degree of curtailment of a solution process is to count the number of steps carried out by the students. Some students may begin with the given formula, others may write a simplification as their first line. The latter case needs to include the implicit simplification in the first line in the line count. In addition, the final form of the solution is often written in a conventional manner, and when a student writes a solution which is not yet in this form, a note should be made that to attain the canonical form to be comparable with other students may require one (or more) further steps.

The following tables show typical solutions of the integration problem

$$\int \sqrt{3x^3}\, dx.$$

for the number of steps given in each column. (Each column may represent slight variants, but the most common solution is written out.) Some solutions do not end in the conventional form $\frac{2\sqrt{3}}{5}x^{5/2}+c$, so these could be considered as requiring one more step to attain standard form for the sake of comparability.

Grade A students all responded correctly and their solutions vary in length from two to six steps (the latter possibly being equivalent to seven steps if the last line were further simplified to its conventional form). (Table 1.)

Typical solutions of grade A students				
All responses correct (12)				
1 student	2 students	5 students	2 students	2 students
$\int\sqrt{3x^3}\,dx$ $=\sqrt{3}\frac{x^{5/2}}{5}(2)+c$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int\sqrt{3}x^{3/2}dx$ $=\sqrt{3}\frac{x^{5/2}}{5}(2)+c$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int(3x^3)^{1/2}\,dx$ $=\sqrt{3}\int x^{3/2}\,dx$ $=\sqrt{3}\left[x^{5/2}.\frac{2}{5}\right]+c$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}(x^3)dx$ $=\sqrt{3}\int x^{3/2}dx$ $=\sqrt{3}\left[\frac{x^{3/2+1}}{3/2+1}\right]$ $=\sqrt{3}\left(\frac{2}{5}\right)x^{5/2}$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{1/2}dx$ $=\int 3^{1/2}(x^3)^{1/2}dx$ $=3^{1/2}\int x^{3/2}dx$ $=\sqrt{3}\frac{x^{5/2}}{\frac{5}{2}}+c$ $=\sqrt{3}\times\frac{2}{5}x^{5/2}+c$ $=\sqrt{3}\frac{2}{5}x^{5/2}+c$
2 steps	3 steps (including unwritten first line)	4 steps (one solution non -conventional)	5 steps	6 steps Both in non -conventional form

Table 1: Grade A student responses to an integration problem

Grade B students produced many errors, with five correct and seven incorrect solutions. Amongst the correct responses, three used four steps and two used six steps. (Table 2.)

Typical solutions of grade B students				
Correct responses (5)		Errors (7)		
3 students	2 students	2 students	3 students	2 students
$\int(3x^3)^{\frac{1}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\left[x^{\frac{5}{2}}.\frac{2}{5}\right]+c$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 3^{\frac{1}{2}}x^{\frac{3}{2}}dx$ $=3^{\frac{1}{2}}\int x^{\frac{3}{2}}dx$ $=3^{\frac{1}{2}}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)+c$ $=3^{\frac{1}{2}}.\frac{2}{5}.x^{5/2}+c$ $=\frac{2}{5}\sqrt{3}x^{5/2}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{1/2}dx$ $=\frac{(3x^3)^{3/2}}{3/2}$ $=\frac{2(3x^3)^{3/2}}{3}$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ Let $u=3x^3$ $\frac{du}{dx}=9x^2$ $dx=\frac{du}{9x^2}$ $\therefore\int(u)^{\frac{1}{2}}dx=\int u^{1/2}\frac{du}{9x^2}$ $=\frac{1}{9x^2}\int u^{1/2}du$ $=\frac{1}{9x^2}\left[\frac{2}{3}u^{3/2}\right]+c$ $=\frac{2u^{3/2}}{27x^2}+c=\frac{2(3x^3)}{27x^2}$ $=\frac{6x^3}{27x^2}=\frac{2}{9}x$	$\int\sqrt{3x^3}dx$ $=\int 3x^{3/2}dx$ $=3\int x^{3/2}dx$ $=3x^{5/2}.2/5$ $=6/5\,x^{5/2}+c$
4 steps	6 steps	Overgeneralisation of integration	Mixture of substitution and direct integration	Algebraic Misconception

Table 2: Grade B student responses to an integration problem

Grade C students have only four correct solutions but one has only 2 steps, one has 3 steps and two have 4 steps. (Table 3.)

From these solutions of students in grades A, B, C we note that the higher attainers in grade A are all successful but vary considerable in the number of steps taken. Grade B students are less successful (5 out of 12) and the correct solutions vary from 4 to 6 steps. The grade C students are even less successful (4 out of 12) and the four successful students have solutions varying in length from 2 to 4 steps. It cannot be asserted that there is any clear pattern between curtailment and attainment. However, there is a clear diminution in lower attaining students successfully completing the problem. The difference between the performance of Grade A and Grade B is statistically significant using the χ^2-test with Yates correction ($p<0.01$), and between Grade A and Grade C even more so ($p<0.0025$). The zero entry in the Grade A failures greatly biases these results, nevertheless the differences are clearly striking.

Typical solutions of grade C students					
Correct responses (4)			Errors (8)		
1 student	1 student	2 students	3 students	2 students	3 students
$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\sqrt{3}\frac{2}{5}x^{5/2}+c$	$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\frac{\sqrt{3}x^{5/2}}{5/2}+c$ $=\frac{2}{5}\sqrt{3}x^{5/2}+c$	$\int(3x^3)^{1/2}dx$ $=\sqrt{3}\int x^{3/2}dx$ $=\sqrt{3}\left[x^{5/2}.\frac{2}{5}\right]+c$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x)^{3/2}dx$ $=2/3(3x^3)^{3/2}+c$	$\int\sqrt{3x^3}dx$ $=(3x^3)^{1/2}$ $u=3x^3$ $=\int(u)^{1/2}$ $=\frac{u^{3/2}}{3/2}+c$ $=\frac{2}{3}u^{3/2}+c$ $=\frac{2}{3}\left(3x^3\right)^{3/2}+c$ $=2x^{9/2}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{3/2}dx$ $=9\int(x^3)^{3/2}dx$ $=9\left[\frac{(x^3)^{1/2}}{\frac{1}{2}}\right]+c$ $=9\left[\frac{(x^3)^{9/2}}{\frac{9}{2}}\right]+c$ $=2[(x^3)^{9/2}]+c$
2 steps Non-conventional solution	3 steps	4 steps	Over-generalisation of direct integration	Mixture of substitution and direct integration	Algebraic Misconception

Table 3: Grade C student responses to an integration problem

Conceptual Preparation

When the manipulation involved in using an algorithm becomes more complex, it may be possible to devise alternate methods to simplify the solution For example, the problem to determine the derivative of $\frac{1+x^2}{x^2}$ using the standard algorithm for the derivative of a quotient involves the student needing to use the formula in a cumbersome way and then simplifying the result:

$$y=\frac{1+x^2}{x^2},$$

$$\frac{dy}{dx}=\frac{(2x)(x^2)-(2x)(1+x^2)}{(x^2)^2}=\frac{2x^3-2x-2x^3}{x^4}=-\frac{2x}{x^4}=-\frac{2}{x^3}$$

However, if the expression is first simplified as $x^{-2}+1$ then its derivative is straight away seen to be $-2x^{-3}$, affording a considerable reduction in processing. Students may shorten their solutions in various ways, for instance, the initial simplification might be conceived as a succession of formal manipulations:

$$\frac{1+x^2}{x^2}=\frac{1}{x^2}+\frac{x^2}{x^2}=x^{-2}+1.$$

However, often students compress this further to a single written step:

$$\frac{1+x^2}{x^2}=x^{-2}+1.$$

Some do this by reading the symbol $\frac{1+x^2}{x^2}$ as two fractions in this way:

$$\frac{1}{x^2} \quad + \quad \frac{x^2}{x^2}$$

translating $\frac{1}{x^2}$ immediately as x^{-2}, then writing $\frac{x^2}{x^2}$ as +1, to perform the simplification in a single composite step.

Out of thirty six students, twenty of them simplified the expression $\frac{1+x^2}{x^2}$ before carrying out the differentiation, for example by writing:

$$y=x^{-2}+1,$$

$$\frac{dy}{dx}=-2x^{-3}=\frac{-2}{x^3}.$$

Fifteen students failed to conceptually prepare and so led to a more complex version of the algorithm and the need to perform more simplification afterwards. All but one student were successful in this task, the remaining student making a single slip by writing a '+' sign in the numerator of the quotient algorithm instead of a '–' sign:

$$\frac{dy}{dx}=\frac{2x(x^2)+2x(1+x^2)}{(x^2)^2}=\frac{2x^3+2x+2x^3}{x^4}=\frac{4x^3+2x}{x^4}=\frac{4}{x}+\frac{2}{x^3}.$$

The students in the various grades performed as follows:

Students' grade	Conceptually prepared	Post-algorithmic simplification	No further simplification
A	10	2	0
B	6	6	0
C	4	7	1
Total	20	15	1

Table 4: Student responses to a differentiation problem

Here the number carrying out conceptual preparation reduces from 10 out of 12 in grade A to only 4 out of 12 in grade C. Using a χ^2 test with Yates correction, this is significant at the 5% level (with p=0.038). The numbers involved are small and the differences between groups A and B and between B and C are not statistically significant.

The fragility of conceptual preparation

The conceptual preparation for a solution depends very much on the nature of the problem. There is no obvious algorithm to cover all possible cases. For instance the derivative of $y = \frac{1+x^2}{x^2}$ is simplified by separating the expression into two parts, but the derivative of

$$y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2}$$

is found more easily by adding the two expressions together and factorising the numerator.

$$y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2} = \frac{1-x^4}{1+x^2} = \frac{(1-x^2)(1+x^2)}{(1+x^2)} = 1 - x^2,$$

$$\frac{dy}{dx} = -2x.$$

In this example, only six of the twelve Grade A students added the terms together and factorised the numerator. Conceptual preparation therefore varies considerably from case to case and is not given by a single algorithm, so students may use some form of conceptual preparation in some problems, but not in others.

Sometimes it may not even be clear whether some form of conceptual preparation may be advantageous. For instance, the problem

$$\text{Find } \frac{dy}{dx}\text{, when } y = \left(x + \frac{1}{x}\right)^n$$

is best solved by using the chain rule with $u = x + \frac{1}{x}$ to obtain the derivative in the form $nu^{n-1}\frac{du}{dx}$. However the problem

$$\text{Find } \frac{dy}{dx}\text{, when } y = \left(x + \frac{1}{x}\right)^2$$

happens to be easier by expanding the bracket to differentiate $x^2 + 2 + x^{-2}$. In this case there is a tension between using the generalisable chain rule method and the particular method expanding the bracket, which happens to be marginally shorter. This is reflected in the performance of the grade A students where six used the chain rule and six expanded the bracket. In interview, four of the six using the chain rule could see a possible advantage in the alternative method but preferred to use the more general strategy and trust their facility in manipulation.

Conclusion

In the group of students studied (between the 50th and 90th percentile in the whole population) there is no obvious correlation between the number of steps taken in carrying out a routine symbolic algorithm and the level of attainment of the student. Thus the curtailment spoken of by Krutetskii in higher attaining children solving problems does not occur here. The more successful Grade A students include those who write out algorithms in greater detail as well as those who curtail the solution. The most obvious difference between the Grade A and Grade C students is the ability of the former to complete the procedure correctly.

However, when problems are designed which can be simplified by an initial conceptual preparation, the more successful students are more likely to conceptually prepare than the less successful students. With problems where the preparation involves using a more specific method that is shorter or a generalisable method which happens to be longer, the more successful students are likely to be aware of the alternatives, some using the shorter method, some preferring the more general method and having confidence in their ability to carry out the manipulation. This is in accord with the notion of proceptual thinking in arithmetic (Gray & Tall, 1994) where the more successful select appropriate conceptual methods or have the power to carry out the procedures correctly. It is also in accord with the value of having both conceptual and procedural knowledge (Hiebert & Lefevre, 1986).

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USING SMALL GROUP DISCUSSIONS TO GATHER EVIDENCE OF MATHEMATICAL POWER

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Abstract

Four grade nine students discussed their solutions to seven mathematical problems. The discussions were analysed to provide evidence of mathematical power which was defined in terms of NCTM's student assessment standards (SAS) and their integration. Results showed that students demonstrated mathematical power to the extent that at least one category of the mathematical activities associated with each SAS was reflected by the students' small group discussions. Combining students' written scripts with their talk provided a better insight into the things they were talking about. Also, monitoring the students and sometimes providing them with prompts helped them to accomplish their tasks. Finally, the students tended to shift their viewpoints consensually or conceptually to align their viewpoints with majority viewpoints.

Introduction

A major reform in mathematics education throughout North America, initiated by the National Council of Teachers of Mathematics (NCTM), involves the provision of standards for curriculum and evaluation in K-12 mathematics (NCTM, 1989) and standards for teaching K-12 mathematics (NCTM, 1991). The standards for curriculum and evaluation, and those for teaching, are the ones perceived by the NCTM as important and needing implementation if students are to develop mathematical power which refers to "all aspects of mathematical knowledge and their integration" (NCTM, 1989, p. 205). Specifically for this study, mathematical power was defined in terms of student-assessment standards (SAS), which comprise mathematical communication, mathematical problem solving, mathematical concepts, mathematical procedures, and mathematical disposition. Associated with each SAS are defined categories of mathematical activities (NCTM, 1989).

One way to monitor the development of mathematical power is through the *talk* that can result when students interact (as can occur in small groups) to make sense of mathematical activities. There have been several investigations into the contributions of student interactions (in small groups) to the learning of mathematics (Artz & Newman, 1990; Davidson, 1990; Johnson, Johnson, & Stanne, 1990; Webb, 1991; Yackel et al., 1990), but none has been directed specifically at examining the extent to which information from the small group is indicative of students' mathematical power. Furthermore, it was reported in the March 1994 issue of the Journal of Research in Mathematics Education (volume 25 number 2, page 115) by the Research Advisory Committee of the NCTM that:

> Perhaps the most obvious research-related response to the *Standards* is the identification and clarification of the research base for the recommendations contained in the document. The *Standards* document contains many recommendations, but in general it does not provide a research context for the recommendations, even when such a context is available.

So, in line with the aspirations of the Research Advisory Committee, this study sought to provide a research context for using the small group format to gather information indicative of students' mathematical power. The information involved what a group of four students said or wrote down individually as they engaged in student-student interactions to discuss their solutions to mathematical problems.

Theoretical Framework

To initiate and sustain verbal interactions among students, some form of discourse is necessary. This discourse includes the way ideas are exchanged and what those ideas entail (NCTM, 1991). Throughout the discourse, the individual's ways of making sense of things (Davis, Maher, & Noddings, 1990), are influenced by the social interaction that helps the individual to make sense of those things (Bishop, 1985; Yackel et al, 1990, Vygotsky, 1978). Accordingly, where individual students interact to discuss their solutions to mathematical problems, I believe it is important to consider the individual's ways of making sense of the problems and the social interaction among the students, both of which contribute to the generation of knowledge. Thus, the ideas of constructivism and knowledge generation through social interaction, provided a useful theoretical framework for gathering information on students' demonstration of mathematical power in small group contexts.

Method

Design

Four of the 18 grade 9 students of a class were selected to form the focus group of the study. The remaining students were also grouped into fours or fives and they participated in the study but data gathered from them were used only for purposes of triangulation. For each data gathering session, the students attempted to solve the assigned problems individually within 20 minutes and then later discussed the solutions they obtained with their group members for 40 minutes. I urged the students to focus on explaining and giving justifications for the solutions they obtained while they discussed their solutions. Occasionally, I gave students prompts either when they asked for help, or when I found they were stuck in their discussions.

Small group formation

As students' talk was very vital for gathering information for the study, it was desirable to have group members who would communicate with each other, feel comfortable sharing their ideas together, validate their conjectures while others in the group tried to meaningfully criticize those conjectures. Also, according to Webb (1991), for equal number of males and females, achievement does not differ significantly when students work in groups. Accordingly, students for the study were selected based on the following criteria: 1) mathematical ability, 2) ability to talk in a group, and 3) balancing of males and females.

The problems

Although what constitutes a problem varies for each student and that not all the problems could provide information indicative of all of SAS, what was important was that each problem must have the potential for students to engage in sound and significant mathematics as a part of accomplishing the task (Van de Walle, 1994). Furthermore, the problem should provide the students the opportunity to have *something* to talk about. In that regard, I tried to use problem types with which the students were familiar, and indeed, students had a lot to talk about.

Data collection techniques

To gather information from the participants' perspectives (Hammersley & Atkinson, 1991; Patton, 1987), I video-recorded the group's discussions of their solutions to the problems. The remaining groups' discussions were audio recorded. Also, I collected all students' written responses to the problems.

Data analysis

The focus group's discussions were transcribed from the video and then analyzed. Information from students' small group discussions were organized around SAS which served as ***key constructs*** (Fetterman, 1989; Guba & Lincoln, 1991; Hammersley & Atkinson, 1991; Merriam, 1991). The unit of analysis (Merriam, 1991; Yin, 1989) was the information students generated in 40 minutes, as they discussed in small groups, their solutions to each of the mathematical problems. Any inferences or generalizations were not statistical, but rather analytical (Yin, 1989), and they were to "guide but not predict one's actions" (Merriam, 1991, p. 176). To provide "trustworthy" results, efforts were made to ensure the "credibility" and "auditability" of the data and the results (Guba & Lincoln, 1991). For example, after the transcription, each video recording and the transcripts were re-examined together.

From the full transcript of each problem, portions were coded as C1, C2, ...; MP1, MP2, ...; MC1, MC2, ...; PS1, PS2, ...; and MD1, MD2, ...; which represent categories of mathematical activities associated with SAS. For example, in PS4, "PS" refers to "problem solving" while "4" refers to an excerpt that reflects the "fourth" category of students' mathematical activities listed under problem solving, that is, *verify and interpret results*. Then, from all portions of the discussions coded PS4, I selected one excerpt that, in my judgment, best illustrates students' ability to *verify and interpret results*, using the NCTM's definition. Transcripts for the other problems were treated similarly.

The extent to which students demonstrated mathematical power was then provided in terms of the interpretations of the excerpts relating to SAS and the union of those excerpts. What was important here was to provide a holistic picture of students' demonstration of mathematical power within problems and across problems.

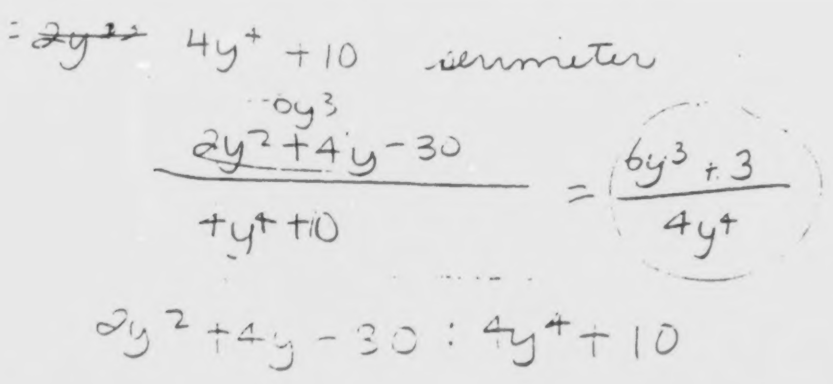
Results and Reflections

A category of mathematical activities associated with mathematical communication (an SAS) was the *use of mathematical vocabulary, notation, and* *structure to represent ideas, describe relationships, and model situations.* An example of students' use of mathematical notations to represent mathematical ideas and to describe mathematical relationships is illustrated by the excerpt in figure 1. Students were discussing their solutions to a problem requiring them to find the ratio of the area to the perimeter of a given plane figure. Jane orally described ratio correctly as "...This, two dots, and that." Her script (see Figure 2) showed that she could represent ratio also with a "slash" instead of a "double dot." When Daniel responded to Jane's question "Does it matter if you write it this way or that way?" by saying that "They are all the right answer", apparently, he was assuring Jane that both ways of representing ratio are correct. Quincy also was in agreement. So, in addition to seeing how students represented notations, we recognize that they also "debated" its appropriateness.

Jane:	So then if it is ratio, it will be like....
Paulina	Is this ratio? [Asking Daniel].
Daniel:	Yes...
Jane:	...This, two dots, and that? Does it matter if you write it this way or that way? What do you think?
Daniel:	They are all the right answer. [Quincy nodding his head].

Figure 1: An excerpt of students' discussions involving mathematical communication.

An examination of Jane's script (Figure 2) shows that "this" was referring to one part of the ratio, "two dots" was referring to the symbol for ratio, and "that" was referring to the other part of the ratio.



$= \cancel{2y^2}$ $4y^4 + 10$ perimeter

$-0y^3$

$$\frac{2y^2+4y-30}{4y^4+10} = \frac{6y^3+3}{4y^4}$$

$$2y^2+4y-30 : 4y^4+10$$

Figure 2: Jane's script

Mathematical power

Evidence of students' demonstration of mathematical power was provided by their: 1) ability to communicate mathematically, 2) ability to use mathematical concepts, 3) ability to use mathematical procedures, 4) ability to use mathematics to solve problems, and 5) disposition towards mathematics. In addition, students' mathematical power was evidenced by the extent to which students integrated all these aspects of what should constitute mathematical knowledge. Table 1 provides a summary of the distribution of excerpts that were reflective of SAS. The table was obtained after analysis similar to the ones in Figures 1 and 2.

Table 1

Categories of Student Assessment Standards (SAS)

Reflected by Discussions of Problems

SAS	Prob 1	Prob 2	Prob 3	Prob 4	Prob 5	Prob 6	Prob 7
C	2/3	3/3	3/3	3/3	2/3	3/3	3/3
MC	2/7	4/7	3/7	4/7	3/7	3/7	2/7
MP	4/7	3/7	4/7	2/7	2/7	3/7	5/7
PS	2/5	2/5	2/5	2/5	2/5	4/5	2/5
MD	4/7	3/7	4/7	4/7	3/7	3/7	6/7

Note:

C = Communication, MC = Mathematical concepts, MP = Mathematical procedures, PS = Problem solving, MD = Mathematical disposition

For example, 2/3 in the row of "C" (communication) and in the column of "Prob 1" (problem 1) means that there were excerpts from the discussions of problem 1 that reflect two out of the three categories of mathematical activities associated with mathematical communication. Notice that 2/3 was not used to mean two out the three *equal* categories; it was only used to mean that two of the three categories were reflected. So, evidence that the discussions reflected mathematical activities associated with any two or more SAS (union of excerpts related to SAS) was taken to constitute evidence for integration. (For further details, see Anku, 1994).

Other results

Two major types of shifts were perceived to have taken place as students discussed their solutions to the problems given them. These were labeled *consensual* when the shift was to align an initial viewpoint with that of the majority, and *conceptual* when an initial conception was abandoned by the students for a different conception during the discussions. Most of the consensual shifts involved majority viewpoints that were compatible with acceptable viewpoints within mathematics. Apparently, students did not shift consensually if they had a solid grasp of an initial viewpoint. Finally, conceptual shifts observed from the study resulted in conceptions that were compatible with standard conceptions within mathematics.

Reflections

There were some difficulties associated with capturing students' mathematical power through the SAS. The circular definition of students' mathematical power made it problematic when deciding what constituted students' mathematical power. For example, the NCTM considers students' mathematical power as one of the student assessment standards and considers mathematical reasoning also as one of the student assessment standards. However, a category of mathematical activity associated with students' mathematical power involves mathematical reasoning also. Thus, conceptually, mathematical reasoning is presented as a subset of students' mathematical power and at the same time presented as of equal importance to students' mathematical power, which is a student assessment standard. What constitutes students' mathematical power was therefore difficult to determine and some conceptual clarification is needed.

Talking about conceptual clarification brings to mind the difficulty I had deciding whether the mathematical power demonstrated by the students in the small group was for the group or for the individuals in the group. During the discussions some particular students seemed to talk frequently, but as responses to what other students, who seemed talk less frequently, said in the group. In either case, the talk reflected a category of mathematical activity associated with one of the students assessment standards. So, was it the student who talked more frequently that demonstrated mathematical power or the one who talked less frequently but who provoked the discussion? Or was it the whole group that demonstrated mathematical power? It was a difficult decision for me to take and I found myself "buying" into the idea that in the small group context, the individual demonstrated mathematical power which was "mediated" by the group interaction. By that I mean there was some "group effect" on the individual's demonstration of mathematical power, and I am still grappling with how to determine the extent of that group effect.

Sometimes, deciding on which categories of mathematical activities particular information reflected was difficult because of the overlap of some of the categories associated with SAS. Evidence that was indicative of a student's ability "to apply a variety of strategies to solve problems", for example, might also be indicative of that student's "flexibility in exploring mathematical ideas and trying alternative methods in solving problems." However, these two categories of mathematical activities are associated with two different SAS. Rather, instead of creating separate categories for such mathematical activities, efforts should be made to unify such categories so as to provide a more holistic picture of students' mathematical power.

Implications and Conclusion

Even though the study could be considered a "best case scenerio," the results suggest several implications for classroom practice. Since the small group discussions provided information indicative of students' mathematical power, the results suggest that the small group context can be used to gather such information. As such, mathematics teachers are encouraged to use it as a context for gathering information indicative of students' mathematical power. Also, mathematics teachers are encouraged to consciously provide for all categories of mathematical activities that are associated with SAS if students are to meet the expectations of the reform. Limiting the categories will limit the extent to which students develop mathematical power. Also, when teachers adopt the use of small groups to gather information indicative of students' mathematical power, they are encouraged not to focus only on students' talk, since sometimes, combining students' talk with their written scripts can provide better insights into the subject of discussion.

A classroom instructional process, which involves discussions of mathematical activities, may help improve students' proficiency in mathematics because as students shift their reasoning consensually or conceptually as a result of group discussions, they tend to align themselves with viewpoints that are compatible with acceptable viewpoints within mathematics. For students to confidently align themselves with acceptable viewpoints, teachers need to encourage their students to self-validate (Anku, in press) their solution. This was evidenced in the study by students not changing their solution when they could self-validate it. Thus, the ability to self-validate should provide the control element shaping the direction of the shifts.

Finally, teachers are encouraged to monitor the group discussions so that prompts can be given to challenge shifts not aligned with acceptable viewpoints within mathematics. Giving the appropriate prompts at the appropriate time means that the teachers are knowledgeable enough to detect students' difficulties (and strengths) and know what prompts to give to help clarify students' thinking. Monitoring is also necessary if teachers are to identify the "buds" or "flowers" that are "in the course of maturing" (Vygotsky, 1978, p. 87) and provide appropriate mathematical activities that will enhance the growth of those *buds* or *flowers*.

In conclusion, this study demonstrates in a small way that from small group discussions, there can be *observable events* that reflect the categories of mathematical activities associated with SAS. To continue with the current reform within mathematics education, teachers should be encouraged to take risks to identify and assess classroom events that reflect the seemingly rhetoric parts of the SAS. Teachers will need a lot of guidance and encouragement, and I hope this study provides an additional source of encouragement that it can be done.

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SEVENTH GRADES STUDENTS' ALGORITHMIC, INTUITIVE AND FORMAL KNOWLEDGE OF MULTIPLICATION AND DIVISION OF NON NEGATIVE RATIONAL NUMBERS

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Abstract

This paper applies a theoretical framework for analyzing seventh grade students' knowledge of rational numbers. A study was designed to examine the possible interrelations among different dimensions of knowledge. Sixty-six Israeli students answered a written questionnaire and were intensively interviewed about their mathematical content knowledge of rational numbers.

A comprehensive picture of the algorithmic, intuitive and formal dimensions of knowledge and the interactions among them is provided. This framework is seen as being potentially extended to other domains of study and other populations.

The non-negative rational numbers constitute a formal extension of the natural numbers. This extension is a major part of the curricula of both upper elementary and middle schools and requires substantial restructuring of the meaning of and operations with numbers. Studies have consistently shown that non-negative rational numbers have long been a stumbling block for many students (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Carpenter, Lindquist, Brown, Kouba, Silver, & Swafford, 1988; Greer, 1994). Each of these studies provided information related to a specific aspect of students' mathematical content knowledge of rational numbers.

Fischbein (1983) has suggested that mathematical knowledge is embedded in a set of connections among algorithmic, intuitive and formal dimensions of knowledge. Usually, the assessment of students' mathematical knowledge considers only the algorithmic dimension. In our opinion, in order to get a comprehensive picture of knowledge and thinking abilities of students, one has to take into account the formal and the intuitive knowledge as well. Proficiency in procedures does not necessarily ensure understanding. Ideally, the dimensions of knowledge should cooperate in the processes of concept acquisition, understanding and problem solving. In reality, though, this is not always the case - often there are serious inconsistencies between students' algorithmic, intuitive and formal knowledge. Such inconsistencies could be the source of common difficulties that

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learners encounter in their mathematical activities, including misconceptions, cognitive obstacles and inadequate usage of algorithms (Fischbein, 1983; Tall & Vinner, 1981; Tirosh, 1990).

The absence of a conceptual framework for analyzing learners' mathematical content knowledge, has made a global understanding impossible. This study therefore assesses seventh graders' mathematical content knowledge of rational numbers in respect to the three dimensions and the interactions among them. We assume that these interactions may better explain students' reasoning when solving a problem, correctly or incorrectly.

Method

Subjects

Sixty-six seventh graders participated in the study. All of them had had formal instruction about operations with non-negative rational numbers (fractions and decimals) during the sixth grade.

Instruments

1. Diagnostic test: The students were asked to complete a diagnostic test which examined their formal, algorithmic and intuitive understanding of rational numbers. The diagnostic test examined the following aspects:
    a) The algorithmic dimension: Ability to compute with fractions and with decimal numbers, and the capability to explain the rationale behind the various algorithmic aspects.
    b) The formal dimension: Ability to identify and give examples of natural numbers, integers and rational numbers. Knowledge related to the hierarchy of subsets of rational numbers and familiarity with the density of rational numbers and with the commutative, associative and distributive laws.
    c) The intuitive dimension: Ability to identify the adequate operations for solving multiplication and division word-problems, capability to produce adequate intuitive models for representing number concepts and operations with them, and competency in evaluating the results of arithmetical operations with rational numbers.
2. Interviews: A sample of 23 students was chosen for extensive interviewing.

    Eleven students were those who showed poor algorithmic, formal and intuitive knowledge, while 12 students had relatively good algorithmic but low formal and intuitive performance. Each subject was interviewed at least three times for 20-40 minutes. The interviews were semi-structured, that is, an interview program was outlined for each subject and additional probes were made during interviews to better understand their conceptions.

Results

A. Algorithmic knowledge.

The students were asked to solve five multiplication problems and nine division problems. Half of the students were asked to solve problems with fractions, the others solved the same problems given in decimal form. Poorest performance was observed on items involving mixed numerical notation in decimals. About 40% of the students did not supply any answer to those items.

Interestingly, dividing 6:11 and 11:6 was problematic. About 20% of the students responded to neither of these items. Twenty percent of the students claimed that it was impossible to divide 6 by 11.

Students' performance on division tasks was poorer than on the multiplication tasks, and their performance with decimals was poorer than with integers and fractions (see Table 1).

Table 1: Seventh Graders' Performance in Algorithmic Tasks (in %)

	Correct Answers			Correct Answers	
	Fractions	Decimals		Fractions	Decimals
0.3·9	97	78	.25:4	55	34
3.75·5	64	67	4:0.25	67	55
9·0.3	88	78	0.2:0.8	64	51
0.75·0.5	88	70	0.8:0.2	79	67
6.25·4.8	64	51	0.2:3	73	42
			0.25:0.6	76	34
			8.25:4.5	52	27
			6.11	53	
			11.6	55	

The students were generally unable to justify the successive steps of the algorithms (their understanding of the algorithms seems instrumental and not relational). During interviews, when asked to explain why a certain algorithm led to its solution, most of the subjects kept repeating the steps in the algorithm. They were surprised that a question such as "Why do you multiply both the divisor and the dividend in 4.5:0.5 by the same number?" could be asked. When asked if it was possible, in a problem like 5.8:2, not to multiply the divisor and the dividend by 10 but to perform the division directly, most of the interviewers argued that "It is impossible, you must first get rid of the decimal point".

B. Intuitive knowledge.

1. *Beliefs about multiplication and division:*

The students were asked to respond to statements related to multiplication and division.

a) In a multiplication problem, the product is always bigger than one.

b) In a multiplication problem, the product is always equal or bigger than both factors.

c) In a division problem, the dividend is always bigger than the divisor.

d) In a division problem, the dividend is always bigger than the quotient.

e) In a division problem, the quotient must be an integer.

f) In a division problem, the divisor can be a fraction.

The vast majority of the students held primitive beliefs concerning the results of multiplication and division. High levels of correct answers occurred only to two items: e (82%), and f (91%). Both were division items which referred to only one magnitude (the quotient or the divisor). Performance on multiplication items referring to only one magnitude (the product) was poor a (53%). Low percentages of correct reactions appeared on the other items, all of which dealt with the relative magnitudes of at least two of the quantities involved in arithmetical expressions c (42%), d (27%), b(29% correct, 26% of whom justified it by multiplication by 0). Students hold common beliefs that "multiplication always makes bigger" and "division always makes smaller"

2. *Representations:*

Most students lacked the ability to construct appropriate representations of operations with rational numbers. About half drew appropriate graphical representations of 1/3, and of 6:2 using mostly either disks or rectangular-region models for 1/3, and set models for 6:2.

The representations of improper fractions and of operations involving fractions were a much more difficult task for the students. Only few (10%) constructed appropriate representations of 3/2, 13% represented 1/3 x 5 in a meaningful way, and about 27% gave appropriate representations of the division expressions (i.e., 4 : 1/4 and 1/4 : 4).

3. *Performance on Word Problems:*

The students were asked to write an appropriate expression for each word problem without performing the computation. They were given four multiplication and seven division word problems.

High levels of correct answers (75% correct) occurred in the multiplication problem involving a natural operator ("A motorcycle needs 0.3 liters of fuel per k.m. How many liters does it need for passing 9 k.m.?"). In this case the operator is a natural number and thus the numerical data in the problem are in accordance with the constraints of the intuitive repeated addition model of multiplication. In a similar problem, albeit with a noninteger operator, the percentage of correct answers was low (35%). In this case 40% of the students chose division instead of multiplication as the suitable expression. This shows that when the numerical data in the problem violate the constraints of the intuitive repeated addition model, this model operates behind the scenes and prevents the right solution (Fischbein, 1993).

Table 2: Performance on Division Problems (in %)

Quotative Model					Partitive Model				
Problem No.	Correct Answer	% Correct	m.i.d.*	No Answer	Problem No	Correct Answer	% Correct	m.i.d	No Answer
1	0.25:0.6	6	20	23	5	6:11	59	-	15
2	0.8:0.2	65	12	11	6	0.25:4	63	9	15
3	4:0.25	58	15	15	7	11:6	58	-	18
4	0.2:3	14	20	24					

* m.i.d- multiplication instead of division

Sixty percent gave correct answers to the partitive division problems and to two out of four quotative division problems (see Table 2). The students had great difficulties on the other two quotative division problems. We believe that in these problems the difficulty of the context exceeded the difficulty caused by the constraints imposed by the division model.

C. Formal knowledge.

Serious deficiencies were identified on the following aspects of formal knowledge:

1. Subsets of rational numbers and their hierarchical structure: Only 33% identified {1,2,3,...} as the set of natural numbers, 13% identified {...-3,-2,-1,0,1,2,3...} as the set of integers, and no more than 28% of the students provided an adequate example of a non-integer, rational number. Only 18% knew that -3/4 is a rational number, and even less (11%) correctly identified 0.251 as a rational number. Students' performance on items which examined their knowledge of the hierarchical structure of the numerical system was also poor. When asked to determine which of the two given sets was a proper subset of the other, 23% correctly agreed that the set of the rational numbers includes the integers, and 48% agreed that

the latter set includes the set of natural numbers. Yet, 39% of the subjects incorrectly agreed that the set of the natural numbers includes the set of integers.

2. The density property of rational numbers: Most students were familiar with the density property of rational numbers. The majority of them (about 60%) provided adequate examples of numbers between two given rational numbers. Likewise, most students could correctly determine whether two given rational numbers were equivalent or not. The only problematic pairs were 4/8 and 35/70 (25% argued that 35/70 was bigger than 4/8 because both the numerator and the denominator in the former were greater than in the latter), and 1/7 and 1.7 (15% argued for equivalency).

3. The commutative, associative and distributive laws: Students were asked to write examples for the use of these laws. About 60% provided adequate examples for the commutative law. Less students (44%) provided adequate examples for the distributive law. Only 20% of the students provided adequate examples of the associative law-the most prominent inadequate examples consisted of two uses of the commutative law, namely, 3 x 4 x 5 = 5 x 4 x 3.

Other items in this category dealt with arguments that were presented as if they were offered by a student, and a subject was asked to comment on it, e.g.: "Roni wrote: 7:(4 + 2) = 7:4 + 7:2 . Is his solution correct? Why?". Forty-three percent correctly argued that 7:(4 + 2) is not equal to 7:4 + 7:2.

D. The interaction between algorithmic and intuitive knowledge.

Students function on different levels in algorithmic and intuitive tasks. Sixty percent of the students who showed adequate algorithmic knowledge, obtaining products smaller than both factors, still claimed that " multiplication makes bigger". Half of the 58% of the students who solved 0.2 : 0.8 correctly, still argued that "the dividend is always bigger than the divisor" and 71% of them claimed that "division always makes smaller". It was obvious that when functioning on the algorithmic level, intuitive knowledge was not considered.

E. The interaction between word problem solving and intuitive knowledge.

a) The repeated addition model for multiplication demands a nayural operator and thus imposes the belief that "multiplication makes bigger". The two multiplication problems on which high levels of performance occurred included natural operators. The other two multiplication problems involved non-integer operators and caused many incorrect answers. For example, the problem:

"There were 9 kg. apples in a store, Yael bought 0.3 of this amount. How many kg. of apples did she buy?" forty percent of the students chose division instead of multiplication as the suitable expression.

We assume that students who know that the answer should be smaller than 9, and hold the intuitive belief that "multiplication makes bigger and division makes smaller", choose division instead of multiplication for solving this problem.

b) 1. The partitive model for division imposes three constraints:

i) the dividend is always bigger than the divisor ii) the divisor must be a natural number, and iii) the dividend is always bigger than the quotient ("division makes smaller").

Only in problems 5 and 6, the numerical data were not in accordance with one constraint of the model (i). Half of the subjects who gave a correct expression for solving these problems, hold the intuitive misbelief "the dividend is always bigger than the divisor". The same percentage of correct expressions was given to problems 5, 7 (6:11, 11:6), although in one the dividend is bigger than the divisor and in the other it is not.

One of our hypotheses was that intuitive beliefs support the performance on solving word problems. Accordingly we expected students who believed that "the dividend is always bigger than the divisor" to change the roles of dividend and divisor in the expression solving these problems. Only few did so.

2. The quotative model imposes only one constraint, namely: "The dividend is always bigger than the divisor". The percentage of correct answers given to the two problems in which the numbers followed the constraints of the quotative model (problems 2, 3) was relatively high (58%, 65%). Low performance percentages (6%, 14%) occurred on the problems in which the dividend was smaller than the divisor. Twenty-five percent of the students did not give any answer to these problems, 25% changed the order of the divisor and dividend (as expected) and almost the same percentage incorrectly chose multiplication (instead of division) as the appropriate operation for solving these problems.

Final Comments

Conceptual understanding of rational numbers is foundational for many vital components of mathematical and scientific reasoning, notably ratio and proportion, algebra and calculus (Greer, 1994). In this paper a new conceptual framework was suggested for analyzing learners' mathematical knowledge of specific mathematical content domains, which takes account of algorithmic, intuitive and formal dimensions and their connections. With this framework, we were able to provide a systematic, comprehensive picture of students' mathematics content knowledge of rational numbers, and the interactions among the different aspects of this knowledge.

It became apparent that students' knowledge of rational numbers is inconsistent. We therefore concluded that, during instruction each component of knowledge should be addressed separately and attention should be paid to integrating the three aspects.

Although we used this framework for analyzing students' knowledge of rational numbers, expansion to other domains seems reasonable. We suggest that the framework be adapted to other populations: students in different grades, to preservice teachers or novice teachers. It can also be extended to other domains of knowledge.

Students' performance on algorithmic tasks indicated a difference between skills on fractions and decimals. The students' performance on fractions was higher than on decimals. They tended to use fractions even when the tasks were given as decimals (e.g., substitute $\frac{1}{4} : \frac{3}{5}$ for 0.25 : 0.6). A possible explanation is that students in Israel learn fractions before decimals, so that their experience with fractions is richer. More research is needed in order to verify this assumption and its educational implications should be considered.

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THEORY AND PRACTICE IN UNDERGRADUATE MATHEMATICS TEACHING: A CASE STUDY

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This paper presents a case study of the beliefs and practices of a university teacher of mathematics. Working in a tradition of practitioner research, the practitioner and an observer have recursively and critically reflected on the practitioner's expressed aims and on the text of a lecture to first year undergraduates. While it was possible to identify ways in which the practitioner attempted to operationalise his aims in structuring the content of the lecture and in his interactions with the students, aspects of his practice were also identified which may have been in conflict with his aims and which suggest tensions between his aims, both for the practitioner himself and for his students.

In spite of increased attention paid by the mathematics education community in recent years to the beliefs and practices of teachers at primary and secondary level (Hoyles, 1992), relatively few researchers have addressed the practices of teachers in higher education. With few exceptions (e.g. Vinner, 1994; Mohammed Yusof & Tall, 1995) PME papers concerned with undergraduate education have focused on students' individual mathematical conceptions rather than on classrooms, teachers or teacher-student interaction. At the same time, however, there is interest in the UK in the quality of university mathematics teaching and, in particular, its response to the needs of undergraduate students with relatively low entry qualifications and to the disjuncture in students' experiences at the interface between school and university mathematics. This interest has been reflected in the establishment in 1992, under the auspices of the Mathematical Association, of a working group on Teaching and Learning Undergraduate Mathematics, bringing together teacher-academics from both Mathematics and Mathematics Education. Arising from the work of this group, we (the authors of the present paper) have undertaken a collaborative project to examine the practice of one mathematics lecturer, the *practitioner* (TB), through an articulation of his own critical reflection on his aims, beliefs and practice with the alternative perspective provided by the *observer* (CM). In this paper, we present a case study of a single lecture, considering the relationships between the lecturer's consciously expressed intentions and his practice, addressing, in particular, the following questions:

- How were the practitioner's aims and beliefs about teaching and learning reflected in teaching method?
- How were his aims communicated to students?
- What areas of possible mismatch were there between the lecturer's and the students' aims and expectations?

Methodology

The study described here is in the tradition of practitioner research concerned with professional 'self-knowledge' and development (Weiner, 1989; Morgan, 1993), examining the lecturer's practice through his own theoretical framework and through the eyes of an informed observer. The interaction between the two participants has been vital in forming the subsequent analysis. It is anticipated that this collaborative enterprise will result in changes in practice (although we do not yet know what these changes will be) and generate further research questions.

We have initially considered a single one hour lecture, towards the end of an introductory first year course in 'Basic Pure Mathematics' taken by a small group of students starting a four year course leading to a degree in Mathematics with Education (preparing them to be mathematics teachers in secondary schools). These students on the whole have somewhat lower initial qualifications in mathematics than those starting a single honours mathematics degree course at the same university. They are thus perceived within the university mathematics department to be likely to find this 'abstract' course difficult. The observer attended this lecture, which consisted of periods of 'question and answer' and of extended exposition by the lecturer, taking notes of what was said by the lecturer and by students and transcribing what was written on the blackboard, relating this to the oral interactions. These notes form the basis of the analysis offered here.

The analysis has been a recursive process, encompassing the interests and perceptions of both participants. Thus each read and wrote a commentary on a section of the notes of the lecture; this text and commentary was then commented upon by the other participant. There were also a number of meetings in which issues arising from the analysis were clarified and points of conflict were discussed and, if not resolved, at least acknowledged and appreciated by both parties. Inevitably, because of our different experiences and perspectives we brought different resources to bear on the analysis, asking slightly different questions and focusing on different aspects. From the practitioner's perspective, the primary focus was on identifying and describing the beliefs and intentions underlying his practice and reflecting on the ways in which these were represented in his own behaviour. The observer's interest lay in a focus on the interaction between lecturer and student, reflecting on the match between the practitioner's expectations, his actions and the ways in which these might be perceived by the students.

Practitioner's theoretical framework and teaching aims

Before looking at some examples from the lecture itself and considering the analysis of these examples, it is necessary to consider the aims and theoretical framework employed by the practitioner. These are expressed in general terms in the statement included in the information sheet given to students at the beginning of the course:

> A general aim of the course is to help students in the transitions from concrete to abstract mathematical thinking and from a purely descriptive view of mathematics to one of definition and deduction.

This statement relates to a model of progression in mathematical development from 'computational', through 'descriptive' to 'deductive' modes of reasoning. This progression is reflected in mathematics curricula and in the performance of students at different stages of an undergraduate course (Barnard, 1995) although it is unlikely to be constant across different topics and contexts. We exemplify it here in the contexts of simultaneous equations and of polynomials:

Computational Numerical and symbolic computations and procedures with a focus on *specific* objects, e.g.

Simultaneous equations: Solving $2x + 3y = 7, \quad 4x - y = 5$.

Polynomials: Differentiating or sketching the graph of $x^2 + 5x + 6$.

This is characteristic of most pre-university mathematics.

Descriptive   Manipulations of more *general* objects and descriptions of general behaviours, e.g.

> *Simultaneous equations*: Describing the various possibilities for the solutions of the system of equations:
>
> $$a_1x + b_1y + c_1z = d_1$$
> $$a_2x + b_2y + c_2z = d_2$$
> $$a_3x + b_3y + c_3z = d_3$$
>
> *Polynomials*: Relating the degrees of the sum and product of the polynomials $a_0 + a_1x + a_2x^2 + \ldots + a_mx^m$ and $b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$ to m and n.

Although the mathematics is largely descriptive, the objects and procedures being described are more general than at the purely computational level. Whereas in computational situations solutions are validated by the computation itself, the increase in generality means that proof now plays a greater role in validating conclusions.

Deductive   Thinking in a more *theoretical* domain of definitions and deductions, in which symbols and words are the predominant features. The behaviour of a system of linear equations is now a theorem about a vector space and the dimensions of certain subspaces. There is also a focus on connections between mathematical structures, such as the fact that the set of integers and the set of polynomials (over a field) both have a unique factorisation property.

The majority of students' mathematical experience before university is of a 'computational' type with the focus on what has to be 'done' in order to achieve a correct outcome. It might be expected, therefore, that these first year undergraduates would experience some difficulty in their transition to descriptive and deductive ways of thinking.

Aims of the case study lecture

The lecture discussed here started with a discussion of the irrationality of $\sqrt{2}$, leading into consideration of polynomials and the theorem:

> If $f(x)$ is a monic polynomial with integer coefficients and $f(x) = u(x)v(x)$, where $u(x)$ and $v(x)$ are monic polynomials with rational coefficients, then $u(x)$ and $v(x)$ must in fact have integer coefficients.

Included as a crucial step in the mathematics was Gauss's Lemma:

> Let $f(x)$ and $g(x)$ be polynomials with integer coefficients. If each of $f(x)$ and $g(x)$ has the property that there is no integer (apart from ± 1) that divides all of its coefficients, then the polynomial $f(x)g(x)$ also has this property.

In discussion with the observer, the practitioner classified his 'content-related' aims in four layers:

Knowledge of the above facts;

Justification, defined as a step by step understanding of the proofs of the results;

Understanding, further subdivided into:

> (a) Holistic understanding of what the theorem is saying and how it fits into a wider picture. For example the theorem carries with it, as a simple special case, the deduction "$\sqrt{2}$ is not an integer implies $\sqrt{2}$ is irrational".

(b) A feeling for *why* the theorem is true and what makes the proof 'tick', for example, the place of the statement "If $f(x)$ and $g(x)$ are polynomials with integer coefficients and p is a prime which divides all the coefficients of $f(x)g(x)$, then either p divides all the coefficients of $f(x)$or p divides all the coefficients of $g(x)$"; and

Culture, looking beyond the particular theorem to a consideration of how the proof reflected certain characteristics of this general area of mathematics. Such characteristics include:

- the feature that although polynomials have lots of 'bits', the bits can often be used like steps in a ladder in order to achieve a proof;
- the closeness of **Z** (the integers) to **Q** (the rationals) in that (a) every element of **Q** is of the form a/b with $a, b \in \mathbf{Z}$ and (b) one can clear denominators of a finite number of fractions;
- the theme that one can often reduce to *primes* because every positive integer is a product of primes;
- the basic property of primes that if p is a prime and a, b are positive integers such that p divides ab, then p must divide either a or b.

It was considered to be useful for the students' future thinking in this area to have these features built into their networks of mental associations.

In the rest of this paper we consider the ways in which the practitioner's aims were manifested in his practice, drawing on extracts from the lecture and the subsequent reflections and discussion between practitioner and observer.

Proof and the move towards deductive modes of reasoning

Examination of the ways in which proof was addressed during the course of the lecture causes us to question the extent to which the practitioner's aim of moving away from the computational towards the descriptive and deductive was reflected in the actuality of the lecture. In building up towards addressing the theorem, the irrationality of roots of prime numbers was revisited. The students had come across the proof of the irrationality of $\sqrt{2}$ in an earlier lecture. The first interaction related to proof was at a general level reflecting the practitioner's 'Culture' aim in relation to mathematics as a whole rather than the specific topic area:

TB $\sqrt{2}$ is irrational; $x^2 = 2$ for no rational x. This is one of my favourite theorems: jump from no integer x to imply no rational x. What's the point of proofs if you believe the theorem already?

S1 *It keeps us in our place.*

S2 *There might be people who don't believe.*

S3 *You can believe in something that's false.*

TB To understand what's going on.

This interaction was not, however, central to the content of the lecture and it must be considered whether the subsequent proof activities during the lecture addressed the stated aim of 'Understanding' or the overall course aim of moving the students towards deductive thinking. Having refreshed the students' memory of the result (but not the proof) for $\sqrt{2}$:

TB Is $\sqrt{3}$ rational or irrational?

S *Irrational*

TB Why?

S *Same as you put* $x^2 = 3$

(written on the board) $$\left\{ \begin{array}{l} \sqrt{3} = \frac{a}{b} \quad (a,b) = 1 \\ 3 = \frac{a^2}{b^2} \\ a^2 = 3b^2 \end{array} \right.$$

TB Now what?

. . . . .

TB We haven't got the even-odd dichotomy to help us so what do we do?

The student's apparently computational orientation in "putting" $x^2 = 3$ as the next step in the proof (i.e. performing the next step in an algorithmic process) is echoed by the lecturer's "Now what?" and "what do we do?". At this stage there is a contrast between the mode of language used by the practitioner and the mode of thinking he intended his students to adopt. While to the practitioner these phrases are seen to be equivalent to descriptive questions such as "what is suggested by this?" or "what can we relate this to?", the students appear to take them as a cue to continue in a computational mode, following a set of procedures that had been previously established in proving the irrationality of $\sqrt{2}$. Even the question "Why?", which might have related to the aim of 'Understanding' established earlier, prompts a response which appears merely to echo the form of the previous proof.

At a later point, however, the lecturer's language changes from this computational mode to ask questions which seem not only to address the aim of 'Understanding' but also to prompt responses that appear to be in a descriptive or deductive mode:

TB a^5 is even. What does it tell us about a?

S *a is even*

. . .

TB . . . we have proved that $\sqrt[5]{2}$ is irrational. In terms of polynomials, $x^5 = 2$, what does that tell us?

S *It might have complex factors.*

Here, the suggestion that a mathematical statement can "tell us" something approaches a deductive mode of thinking in which the next step of the proof arises from the meaning of the previous statement rather than from manipulation of its symbolic form.

In spite of the clarity of the practitioner's aims in relation to deductive thinking, the differences in the language he uses in constructing proofs during the lecture suggest a mismatch or at least a lack of clarity in his practice.

Tensions between aims

The practitioner felt that Gauss's Lemma was a result of a kind that would be fairly new to the students, as it is a 'descriptive' result, saying that if two polynomials each have a certain property, then so does their product. It was also felt that the property in question, the coefficients having no common factor, might have had few links with the students' previous mathematical experience. A numerical example was provided to help the students engage with the meaning of the result. In order to move away from a computational orientation, however, it was necessary to address a proof of the general result. There was a choice here between giving the details of the proof in full (*Justification*) or giving the underlying ideas (*Understanding* and *Culture*). With most theorems in this course the practitioner would do both, believing there to be a link between the two. However, he also believed that, when synthesising a number of components of a proof, there is often a delicate balance between trying to help the students to keep all the components in their mind and trying to help them to consolidate their understanding of each individual component. On this occasion he decided that there was more to gain by trying to cover the essential ideas of the proof rather than all its details, deliberately suppressing 'Justification' in favour of 'Understanding' and 'Culture'.

TB Gauss's Lemma. See if we can understand what it's saying. $f(x)$ is something like $3x+4$ and $g(x)$ is $2x^2-x+5$. There's an additional assumption that no integer divides all the coefficients. . . .

(on the board) $\begin{cases}(3x+4)(2x^2-x+5)\\ =6x^3+5x^2+11x+20\end{cases}$

You know how to do this. [You can see that there's no integer which divides all the coefficients of the product polynomial.][1] But it doesn't prove the general result. You're all dying to know why [the result is true in general].

S *Yeah*

TB And you'll be relieved to know I'm not going to prove it.

S *Awh*

TB Why is it true? This property [that no integer divides all the coefficients] implies that if it is true then no prime number [divides all the coefficients] and conversely if no prime number [divides all the coefficients] then it is true. So we can reduce it to

(on the board) { "There is no prime number which divides all the coefficients"

Take a particular prime number

(on the board) { $p \quad p|ab \Rightarrow p|a$ or $p|b$

We can show using the essence of this:

(on the board) $\begin{cases}p|\text{all coeffs of } f(x)g(x)\\ p|\text{all coeffs of } f(x) \text{ or } p|\text{all coeffs of } g(x)\end{cases}$

With polynomials, the bad news is there are lots of bits. The good news is you can use the bits as steps. With a bit of induction thrown in, this is the essence of a proof of Gauss's Lemma.

The main points contained in this "essence" of a proof coincide with the practitioner's content-related aims of 'Understanding' and 'Culture' elaborated above, including the possibility of reducing a statement about all integers to one about primes and the linking of the property of primes that "*p* divides *ab* implies *p* divides *a* or *p* divides *b*" with the idea that it is possible to use the "bits" of a polynomial as steps.

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[1] The recording of this section of the lecture was incomplete. The phrases in square brackets have been inserted to indicate the sense of the original utterances.

As may be seen in the extract, the lecturer characteristically makes use of a number of informal 'asides' (e.g. "You're all dying to know why.") which not only contribute towards establishing his relationship with the students but simultaneously serve to induct them into mathematical value systems. Thus asking the question "why?" is seen to be important; a numerical example is not enough to prove a result; an exposition of the "essence" of a proof may answer the question "why?" but does not prove in itself.

Similarly, at a number of points throughout the lecture, references were made to aesthetic and affective responses to mathematics. For example:

> If you can't split it up with integer coefficients then you can't with rationals. Do you see how beautiful that is?
> *(Murmur - a little response from students)*
> See the power, even with all those rationals the theorem says you still can't factorise it.

The practitioner is clearly attempting to communicate 'Cultural' aims at a very general level as well as in relation to the specific content of the lecture. Nevertheless, in reflection after the lecture, he was sceptical about the extent to which these aims could be achieved within the context of the course, writing:

> The students would have been mainly focusing on the first two content-related aims [*Knowledge* and *Justification*], perhaps partly due to time sequential ordering (you can't understand what you don't know) and partly for survival type reasons such as the need to understand subsequent parts of the course and to succeed in the examination. However they would have been aware that there *was* something more that the lecturer was trying to convey.

Thus the practitioner himself perceives a mismatch not only between his aims and those of the students but also between his aims and the institutional restrictions of syllabus and examination system. Indeed, this tension is manifested within his own practice. Not only did his introduction to the lecture include the statement "The aim today is to do enough so you can get on with the sheet", but the homework sheet itself consisted entirely of specific examples of polynomials on which the students were expected to operate largely computationally (either to decide whether or not they were irreducible or to factorise them into irreducible polynomials). For the practitioner, the aim to "get on with the sheet" was clearly secondary, merely providing an indication of the amount of material he planned to cover during the lecture. Nevertheless, the degree of success in answering the questions on the sheet was likely to be the main means by which both lecturer and students could evaluate their performance. Of course, the evaluation of 'Understanding' or 'Culture' would be considerably more problematic.

Conclusions

In studying this lecture, we can see that the practitioner's aims were stated explicitly in course documentation and were reflected in a number of general statements (i.e. 'asides' without reference to the specific content of the lecture) made in oral interactions with the students. In his planning of the approach to the proving of Gauss's Lemma the practitioner had made conscious choices between his various content-related aims.

There were, however, some areas in which his practice appeared not to match closely with his overall aim to move from reliance on computational reasoning towards descriptive and deductive reasoning. The language used to guide students through proofs has been identified as one such area; in particular, the use of the phrase "what next?" would seem to encourage a computational approach. Another, perhaps more powerful, area of mismatch lies in the evaluation structure of the course: the homework sheets and examinations. Many of the questions set appear designed to focus students' attention on the more computational aspects of the course and the first two layers (*Knowledge* and *Justification*) of the practitioner's content-related aims. It appears likely that the 'higher' layers of content-related aims (*Understanding* and *Culture*) may best be achieved if the students spend additional time outside the lecture not only working on the problem sheets but also in reflection on the content of their lecture notes and handouts, in further reading, and in discussion of the topic with their colleagues. These 'higher' aims were clearly ascribed value within the lecture; it is, however, unclear how they could be valued by the assessment system and thus become important primary aims for the students. While the students were advised at the start of the course that attending the lectures and completing the problem sheets would not suffice, it may be that such relatively context free advice requires further reinforcement throughout the course and that the practitioner's aims at all levels need to be made more explicit to the students at all stages of the course.

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HEURISTICS AND BIASES IN SECONDARY SCHOOL STUDENTS' REASONING ABOUT PROBABILITY [1]

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ABSTRACT

In this paper the responses of 247 secondary students to 8 test items used in classical studies of probabilistic reasoning (representativeness, equiprobability bias and outcome approach) are analyzed. The study was designed to assess the quality of probabilistic reasoning of two levels of secondary students (14 and 18 year-old students). These groups are compared revealing few differences in their responses.

STUDENTS' INTUITIONS AND TEACHING PROBABILITY

New mathematics curricula for elementary and secondary school are being introduced in Spain as well as in other countries around the world. These curricula reflect a change in beliefs about how probability should be taught. While probability has been included in a limited scope in the secondary schools, typically as part of a mathematics course emphasing computation such as combinatorics, current curricula being implemented introduce probability in earlier grades. Newer approaches suggest an active learning format where students first make predictions about the chance of occurance for different outcomes, then do experiments with random devices such as spinners, dice and coins, record their results, and compare the experimental probabilities generated to their original predictions.

Indeed, several researchers have recommended this method as a way to encourage students to confront and correct their misconceptions about chance events (e.g., Godino et al., 1987; delMas and Bart, 1989; Shaughnessy, 1992). Because students often hold incorrect views about probability and randomness, Garfield (1995) suggests that effective teaching be based on knowledge of students' preconceptions, and that when learning something new, students construct their own meaning by

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connecting the new information to what they already believe to be true.

Konold's research on probabilistic reasoning (1995) suggests that merely having students make predictions and compare these to experimental data is not sufficient to make students to change their conceptions, because enough data are rarely collected to reveal the correct patterns of outcomes, students' attention are limited, and data variability is typically ignored.

Background

According to Kahneman et al. (1982), statistically naive people estimate the likelihood of events by using judgmental heuristics such as representativeness and availability. People using the representativeness heuristic tend to estimate the likelihood for an event based on how well it represents some aspects of the parent population. They tend to believe that even small samples should reflect the population distribution or the process by which random outcomes are generated.

More recent research suggests other possible explanations for people's poor or inconsistant performance on probabilistic tasks. Lecoutre (1992) described an equiprobability bias as a tendency for individuals to think of random events as "equiprobable" by nature, and to judge as equally likely outcomes that occur with different probabilities.

Konold (1989) identified an "outcome approach" to interpreting probablities. People using this approach, when confronted with an uncertain situation, do not see their goal as specifying probabilities that reflect the distribution of occurrences in a series of events, but as predicting the result of a single trial. Research by Fischbein et al. (1991) identified errors in solving probability problems as due to students' difficulties detaching the mathematical structure from the context of a stochastic situation.

This is a brief summary of a wide variety of research studies that document errors in probabilistic reasoning from young children to adults. This research suggests that an approach to teaching probablity based on predictions and experiments may not be enough to help students form correct ideas about probablity and strategies for solving stochastic problems.

Therefore, our study was designed to assess the quality of probabilistic reasoning of two levels of secondary students, those who had not studied probability and those who had studied probability in a traditional, mathematical way. We were interested in the extent to which these students demonstrated normative reasoning or appeared to be solving problems based on use of misconceptions or faulty heuristics, and if differences in responses would be revealed for the two groups of students.

DESCRIPTION OF THE STUDY

Methodology

A questionnaire was administered to 277 Spanish secondary school students in the spring of 1995. About half of the students (n= 147) were in their first year of secondary school (14 years-old) and had not studied probability. The rest of the students (n=130) were in their last year of secondary education (18 years-old, pre-university level) and had studied probability with a formal, mathematical approach for about a month the previous school year.

Questionnaire

The questionnaire (presented in the Appendix) included 8 items that have been used with slight variations in previous research (e.g., Green, 1982, Lecoutre and Durand 1988, Garfield and Delmas 1991, Fischbein et al. 1991, Lecoutre 1992, Konold et al. 1993 and Madsen, 1995). Items were selected to assess whether students had some particular misconceptions or used incorrect heuristics. These types of incorrect reasoning and the corresponding items are described below.

Representativeness

The first two items assess if students were using the representativeness heuristic to judge the likelihood of different sequences of coin tosses. Although, from a normative point of view, all such sequences are equally likely to occur, sequence b may appear more representative than the others. Item 4 tests students' intuitions about binomial probabilities. We expected that students who reason with the representativeness heuristic would choose the correct answer c (although for the incorrect reason).

Outcome Approach

Konold et al. (1993) suggest that some students could obtain the correct answer to item 1 by reasoning according to the outcome approach. Students who understand the idea of independence and equally likely outcomes would select the correct response to both items 1 and 2. Therefore, it is important to contrast the responses to items 1 and 2 together.

Neglect of sample size: law of small numbers

Item 3 is adapted from Kahneman et al. (1982) to assess whether students appear to be neglecting the sample size in judging probabilities. This is a special case of the representativeness heuristic, referred to as the "law of small numbers," because people tend to judge small samples as equally representive of a population as large samples.

Equiprobability Bias

Some of Lecoutre's (1992) items were used to assess whether students tended to reason using the equiprobability bias. For Item 4, response d is more likely to be chosen by students with equiprobability bias. In items 5 through 8 combinatorial understanding is needed to choose the most likely result. The incorrect responses *a* in item 5, *d* in item 6 and 7 and *e* in item 8 may be obtained by the equiprobability bias.

DISCUSSION

In Table 1 we present the percentages of the students' responses to each individual item. Chi-square tests were used to compare the distribution of responses for each item for the two groups of student. The p-values for the Chi-square tests are shown in the table. It is apparent that, in general, students did very poorly on the test, with always fewer than half of the students getting any item correct. The most difficult item for students appears to be item 5, and the next most difficult item was number 3.

Difference in age groups

The older students had a greater percentage of correct answers for items 1, 2, 4 and 5. Nevertheless, there were no significant differences in responses for the two groups of students for items 3, 6 and 7. The younger students had a higher percentage of correct responses for item 8. Probabilistic reasoning appears to increase slightly for older students, which is not surprising given that these students have had some formal study of probability.

Table 1: Percentages of correct answer, main distracter and other responses in the two groups of students

Item	% correct		main distracter		other		p-value
	age14 (n=147)	age 18 (n=130)	age 14 (n=147)	age 18 (n=130)	age 14 (n=147)	age 18 (n=130)	
1	46.3	65.4	b 35.4	b 23.1	18.3	11.5	0.05
2	39.4	53.8	d 23.8	d 20.0	36.8	26.2	0.03
3	23.8	26.9	c 63.3	c 60.0	12.9	13.1	n.s
4	35.4	43.8	d 45.6	d 41.5	19.0	14.7	0.03
5	15.6	19.2	d 62.6	d 52.3	21.8	28.5	0.04
6	22.4	23.1	e 46.6	e 43.1	32.0	33.8	n.s
7	40.1	50.8	d 36.7	d 36.2	23.2	13.0	n.s
8	36.7	30.0	e 39.5	e 56.2	23.8	13.8	0.01

Types of Misconceptions revealed

Although 55.2% of students gave the correct answer to item 1, the percentage of students who gave correct answers to items 1, 2 and 4 was only 9%, which suggests that very few students in either age group use normative reasoning to answer these probability problems.

A larger percentage (42%) gave correct answers to both of the first items which involved coin toses, while 24% of those who gave the correct answer to item 1 gave an incorrect response to item 2, possibly indicating an outcome orientation, a result also noted by Konold et al. (1993) in their previous study. However, the percentage of students who changed their response was higher in Konold et al. research than in our study.

The percentage of students who selected as the response *b* to items 1 and 2, and gave a correct answer to item 4 was 22%, possibly suggesting their reasoning according to representativeness heuristics.

Based on response to item 3, most students (62,7%) appeared to judge the large sample to be just as representative as the small sample, and that both hospitals were equally likely to have 80% or more boys on a particular day. This suggests the "law of small numbers" aspect of the representativeness heuristics.

Students' difficulty with items 5 and 6 did not necessarily represent the equiprobability bias, because a large group of students (57% in item 5 and 44% in item 6 selected the response: "it is impossible to give an answer", possibly indicating an outcome orientation, while only 18.1% in item 5 and 26% in item 6 chose answer *a* which could reveal the equiprobability bias.

However a large number of students appeared to use the equiprobability bias responses in items 5 (18.1%), 6 (26%), 7 (36.5%) and 8 (47.6%). Only 13 students (5%) gave correct response to all four of these items.

CONCLUSIONS

All the items in our study asked students to compare the likelihood of different events associated with random experiments consisting of more than one trial. These items have been taken from different studies in which students' incorrect responses have been used to develop theories about patterns in probabilistic reasoning.

Our results support previous research, suggesting that studenťs have great difficulty in using probabilistic reasoning and appear to use othes types of heuristics to solve basic probability problems, even after formal mathematical instruction on the subject. While our data support some of the previous studies on misconceptions, they raise new questions about the role of students' prior knowledge and reasoning as they receive instruction in probability.

As pointed out by Godino and Batanero (in press) some learning misconceptions and difficulties cannot be just explained by mental processes, but by recognizing the complexity of mathematical objects and the necessarily incomplete teaching processes in schools. Consequently, we recommend further research into students' probabilistic reasoning, as an essential step for selecting teaching and assessment situations. We hope to address these issues in future studies.

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APPENDIX: QUESTIONNAIRE

Item 1

Which of the following sequences is more likely to result from flipping a fair coin 5 times? a) HHHTT; b) HTTHT; c) THTTT; d) HTHTH; e) All four sequences are equally likely.

Item 2

Which of the above sequences would be least likely to occur?

Item 3

In a certain town hospital a record of the number of boys and girls newborns is kept. Which of these cases is more likely:

a) There will be 8 or more boys in the following 10 newborns.
b) There will be 80 or more boys in the 100 following newborns.
c) Both a) and b) are equally likely.

Item 4

If we observe the following 10 newborns, which of these things is more likely to you?

a) the fraction of boys will be greater or equal to 7/10.
b) the fraction of boys will be less or equal to 3/10.
c) the fraction of boys will be included between 4/10 and 6/10.
d) All these are equally likely.

Item 5

When two dice are simultaneously thrown it is possible that one of the following results occurs: Result 1: 5 and 6 are obtained; Result 2: 5 is obtained twice. Select the response that you agree the most:

a) The chances of obtaining each of these results is equal.
b) There is more chance of obtaining result 1.
c) There is more chance of obtaining result 2.
d) It is impossible to give an answer.

Item 6

When three dice are simultaneously thrown, which of the following results is most likely to be obtained?

a) Result 1: A 5, a 3, and a 6. b) Result 2: A 5 three times.
c) Result 3: A 5 twice and a 3. d) All three results are equally likely.
e) It is impossible to give an answer.

Item 7

Is some of the previous events in item 6 least likely to be obtained?

Item 8

A spinner is divided in 5 equal sized areas numbered from 1 to 5. Which of the following results is more likely to result from the spinner 3 times?

a) 2,1,5 in this exact order. b) 2,1,5 in every order.
c) 1,1,5 in every order. d) a) and b) are equally likely.
e) a), b) and c) are equally likely.

CHANGES IN STUDENT TEACHER VIEWS OF THE MATHEMATICS TEACHING/LEARNING PROCESS AT THE SECONDARY SCHOOL LEVEL

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Within the framework of a mathematics teacher-training program, a variety of teaching strategies are used with the student teachers with the objective of changing their views of mathematics teaching. A study was conducted on one group of students entering the training program and on another group graduating from the same program in order to identify how their views evolve during the program. The changes identified indicate a shift in their perception of the mathematics teacher from that of a skillful communicator seeking to transmit his or her passion and knowledge to others to that of a teacher concerned primarily with initiating a learner-centered interactive thought process that actively engages both teacher and student and whose starting point is the students' knowledge and errors.

Introduction

Studies of teacher-training programs indicate that student teachers are generally offered little opportunity to change the views they formed of mathematics and how it is taught and learned during their years of pre-university schooling (Kagan 1992). These studies also reveal that in their own classroom practices and teaching strategies, student teachers consistently adhere to their previously acquired views. When faced with a problem in the classroom, they tend to resort very quickly to a certain "habitus" (Bourdieu 1980) that stem automatically from their own 13-odd years of experience as students.

> The winner in the young teacher's conflicting situations then is the reliably rooted habitus from one's own experiences as a student. Through these typical regressions, the functioning of this kind of habitus readily supports the old "solutions" and the reproduction of the old school (Bauersfeld 1994, p. 179).

As studies have shown—particularly those of classroom culture (Bauersfeld 1980; Voigt 1985, 1989) and of the *"contrat didactique"* (Schubauer Leoni 1986, 1988) —certain views underlie student teachers' approaches to specific knowledge. By examining student/teacher interactions, these studies in fact help shed light on the system of reciprocal expectations at work in the specific situations studied. They show the strong influence of teachers' prior experiences with the given problem and their own gradually acquired ideas of the teaching/learning process on the strategies they adopt for use with their own students in similar problem situations.

If teacher-training programs are to effectively counterbalance student teachers' socialization experiences during their 12 to 13 years of prior schooling, they cannot overlook the fact that these students bring with them their own previously formed views of mathematics and how it is taught and learned. This raises the question of how to bring about the necessary changes in the way student teachers view mathematics teaching which will be fundamental in their future practice. It is with this in mind that we developed our secondary-level mathematics teacher-training program, the principles and content of which will be discussed in a later section of this paper.

An initial analysis conducted of the student cohort entering the teacher-training program at the Université du Québec à Montréal (UQAM) was part of a broader study whose goal was to delineate more clearly the changes that take place in student-teacher views at key stages in the training. The analysis conducted at this stage focused on the students' entry profiles prior to training, and included a comparison of their profiles with those of a group of students graduating from the same program.

The objectives of this stage of the research project were as follows:

- to identify the views held by students entering the teacher-training program with regard to mathematics and how it is taught and learned;
- to develop a clearer understanding of how the initial views held by student teachers evolve during the program.

The Teacher-Training Program: Principles and Content

The approach used in our teacher-training program, which is based on a socio-constructivist perspective, was developed around the reasoning processes and ideas of student teachers, and was designed to encourage them to evolve in their ways of thinking. The initiatives taken to achieve this end are varied, and involve aspects such as mathematical training within the program (workshop for exploring mathematical activity, and courses focused on mathematical activity that involve areas such as numerical structures, geometry, probability, and statistics). An epistemological thought process is promoted in all courses, particularly in the course on the history of mathematics. In an interrelated way, the initiatives also involve teaching training by means of courses of didactic (didactic and labs that focus on the content of the first cycle of secondary school, proportional reasoning and related concepts, algebra, variables and functions, measurement, etc.) and teaching practicums.

All aspects of the program give priority to student participation in activities in which asking questions, explaining different points of view, and teacher/student interaction play an important role. The training itself is organized as a culture that implements the strategies being taught so as to encourage participants to develop "mathematical habits" different from those acquired during their own years as students.

> To understand the mathematics teaching/learning process, the culture-participation model would appear to be more pertinent than the knowledge-transmission model, or one that introduces students to a body of objective knowledge. Participating in the mathematical process in the classroom also means participating in a culture that uses mathematics or, better still, in a

> culture of "mathematization"(Bauersfeld 1994, p. 177). [translation ours]

One of the main objectives of the initial mathematics course, that student teachers in the program are required to take, is therefore to expose them to an approach to mathematics that is different from the one they have experienced previously, and to introduce them, in a problem-solving context, to another mathematics-teaching culture by instituting a true process of explanation, discussion, and negotiation within the classroom. A number of problems that differ in form and content are used to initiate the process. Working in teams, the students identify each problem and endeavour to solve it. They present a variety of possible solutions to the class immediately after completion of the teamwork. At this stage, they are encouraged to verbalize their strategies and their reasoning. Different solutions are compared and supporting arguments provided. The student teachers' personal views regarding mathematics and how it is taught and learned are indirectly called into question through these discussions.

Other activities in the training program are geared more specifically to the teaching of mathematics. The concerns of the teacher in the classroom setting are a focal point of the didactic courses, and are therefore approached in the following ways: (1) To develop insight into students and their difficulties, reasoning processes, viewpoints, etc., actual student work is used, together with a bank of student errors, and videos of their oral participation and actions in both interview and classroom situations; the student teachers are also given the opportunity to question and observe students. (2) The student teachers are prepared to diagnose students' procedures in real situations. They are asked to analyze students' errors and reasoning processes and to develop strategies for dealing with these errors. (3) Situations are proposed, implemented, analyzed, and queried in such a way that the student teachers learn to choose those scenarios that are pertinent to conceptual learning. (4) The student teachers are continually called upon to verbalize their mathematical reasoning or ideas. (5) Each student teacher is repeatedly called upon to plan lessons and teaching sequences on the basis of his own conceptual analysis of the notion(s) to be covered. This includes trying to anticipate student difficulties and reasoning, and to develop classroom strategies that they will later try out and reconsider. Thus these future teachers are gradually trained to "reflect in and on action" (Schön 1983, 1987). The different activities offered in the context of this three-year training program, as outlined in this paper, give the student teachers the knowledge they require to teach in a way that will enable their students to participate actively in the process of developing their own knowledge.

The aim of the study described in this paper was twofold: first, to develop a better global understanding of the potential impact of this training program and its limitations; and second, to lay the groundwork for a subsequent, more in-depth study of the particular aspects of this training that help catalyze changes in the views and practices of future teachers. The second study will be a developmental study to be conducted on a student cohort that will be followed for a three-year period. It will be combined with an analysis of different training situations.

Methodology

A questionnaire was administered to two groups of students enrolled in the secondary-level mathematics teacher-training program. The first group included 71 beginning students, while the second group included 51 students who were in the third year of the program and had completed their didactic courses and their practicums. The questionnaire had five main parts, which included statements that the students were asked either to rate on a scale of 1 to 5 (total disagreement to total agreement) or to put in order of priority. The first part of the questionnaire dealt with mathematics and was designed to pinpoint the students' underlying views on the subject itself (mathematics as a human construct that is, or is not, part of a social context, versus mathematics as a pre-existing, independent body of knowledge). A number of specific items were designed to assess the value that the students place on reasoning, proof, validation, definitions, subject-specific language and vocabulary, the use of symbols, representations, and concrete material in mathematical activities. The second part of the questionnaire had to do with the learning of mathematics and was designed to elicit the students' views on the topic (learning as a construct of the student, whether or not it fits into a social context, versus learning as the process of imitating given models). Certain items were designed specifically to elicit the respondents' views on the role of manipulation and error in this learning process. The third part of the questionnaire dealt with the teaching of mathematics (teaching that fosters an interactive thought process and takes the student into account, versus teaching as the passing on of pre-determined knowledge). Even more specific, the fourth part of the questionnaire sought to identify the objectives that the student teachers consider relatively important in the mathematics teaching process, and the fifth part, the pedagogical practices which, a priori, they deem valuable. The final part of the questionnaire consisted of an open-ended question in which the students were asked to express their views on the following question: What do you consider to be the characteristics of a good mathematics teacher?

A few individual or focus-group interviews were also conducted with students from both groups (beginners and graduating students) as a means of gleaning additional information and furthering analysis of the questionnaire results.

Profile of Students Entering the Program

Our analysis of the students' answers to the questionnaire (*cf.* Table 1) reveals a somewhat mechanistic view of the problem-solving process in mathematics: in fact, even though respondents agree that there are always several ways of solving a mathematical problem (# 4), that students must be encouraged to think along these lines (# 25), that problem solving involves much more than simply finding a solution to the problem (# 6) and in fact implies creativity (# 10), they nonetheless perceive the problem-solving process as a matter of applying computational rules

(# 9) and as a highly procedural activity (# 15), to which the school system often limits it. The importance that student teachers place on exercises (# 30) and on the learning of mathematical formulas and algorithms (# 36) is very revealing in this regard. The student teachers entering the program attribute considerable importance to definitions in mathematics, which they deem essential to know (# 3), to the use of symbols (# 8), and to the language and vocabulary specific to mathematics (# 12). This firmly engrained perception may well have major repercussions on the way they teach, as can be seen by the practices they favour in their comments on this topic (# 63). A certain open-mindedness was nevertheless noted in the new students which could be built upon during training when dealing with the role of manipulation, materials (# 5, 20), and representations (# 11 and 62) in learning mathematics, and the importance of helping the learner develop reasoning skills (# 32). The view that these beginning student teachers have of the way students learn mathematics totally disregards the key role played by the learner in the process: from the student teachers' point of view, the student learns by imitating a certain model (# 21). Few of them think that students come to school with any knowledge on which new learning can be built (# 19), nor do they see the majority of learner errors as having any logic behind them; rather, they attribute these errors mainly to carelessness (# 18).

Our analysis of the objectives and practices automatically favoured by these student teachers further indicates a factor that is confirmed in their answers to the open-ended question, namely, a concern with showing the usefulness of mathematics (# 33), motivating their students to like the subject, fostering inquisitiveness about the subject (# 35, 44), and bringing mathematics within reach of their students (# 55, 61). Obviously in this sense they are concerned about the student per se, and they are generally in agreement with the questions concerning the student. However, given their view of the learning process, we are still left wondering what they actually perceive the student's role to be in this process. They seem to focus on the teacher's qualities and on his or her manner of presentation, which they feel must be stimulating, clear, well organized, logical and accurate if it is to be within the students' reach and pique their curiosity.

Table 1 — Average ratings given by each student group for specific items on the questionnaire (B: beginners, G: graduating students)

Questionnaire Items	B	G
About mathematics		
3 Mathematics is based on a set of definitions that must be learned.	**3.44**	**2.27**
4 There are always several ways to solve mathematical problems.	3.89	4.31
5 Exploring situations by means of concrete materials is not doing mathematics.	**2.23**	**1.31**
6 Solving mathematical problems means finding the right answers.	2.54	2.03
8 Without symbols, there is no mathematics.	**3.46**	**1.76**
9 Solving mathematical problems means applying computational rules	**3.31**	**1.98**

11 Exploring a situation by means of drawings or diagrams in order to find a solution is not doing mathematics.	1.54	1.27
12 Without a specific language and vocabulary, there is no mathematics.	**3.07**	**2.09**
15 There is always a rule to follow when solving problems in mathematics.	**3.52**	**2.21**
About learning		
18 Most student errors in mathematics are due to carelessness.	**3.47**	**2.68**
19 When children begin school, they have everything to learn.	**2.72**	**1.84**
21 Students learn mathematics by following a model presented by the teacher.	**3.72**	**2.5**
23 Students cannot discover mathematical concepts and principles on their own.	**2.22**	**1.58**
20 Exploration and manipulation are relevant only for early learning activities (numbers, operations) and young children.	1.8	1.47
About teaching		
25 Students should be encouraged to find more than one way of solving a problem.	4.68	4.54
30 Exercises develop skills. Students should therefore be given a lot of exercises.	4.23	3.23
31 Teachers should assume that students do not have enough knowledge to discuss mathematical concepts in an exploration activity.	2.06	**1.72**
32 Teachers should provide the students with the basic knowledge that will allow them to reason.	4.75	4.49
33 Teachers should make sure that the students acquire the basic skills needed in everyday life.	4.37	4.21
35 It is important to foster inquisitiveness in the students.	4.72	4.56
36 It is important that the students be taught mathematical formulas and algorithms.	**3.52**	**2.31**
44 Students should be taught in such a way that they enjoy mathematics.	4.72	4.6
55 Teachers should avoid using symbols too soon.	**3.57**	**4.31**
61 Teachers should use mathematical language that is within the students' grasp.	4.85	4.68
62 A variety of representations should be used to present a topic.	4.43	4.35
63 Symbols should be used at each stage of the teaching process.	**3.87**	**2.56**

Lastly, our analysis of the respondents' answers to the open-ended question confirms and clarifies the views that the new student teachers have of what makes a good mathematics teacher. Whether they speak primarily of arousing curiosity, interest or motivation, of accessibility of teaching style, of making the subject matter intelligible to the students or developing reasoning skills, one overriding idea prevails—that of transmission: "transmitting in the most appropriate language, getting the subject matter across..., knowing how to transmit..., transmitting their passion for mathematics..., transmitting mathematical reasoning skills..., transmitting the skill of comprehension..., transmitting the desire to learn mathematics..., communicating and transmitting their interest...," and the list goes on. If the student is mentioned in their initial concerns, it is often in terms of the teacher's availability to the student. They state that the teacher must be "patient, available to answer questions outside of class hours, be prepared to repeat the same problem twenty times until the students understand, repeat explanations until everything is clear for the students, encourage them and give them confidence, etc". These comments focus on the way in which the subject matter is communicated; the teacher must be "clear, well organized, accurate, understandable; he must communicate in a way that is easily understood; and he must be articulated, know how to communicate in a dynamic manner, have a thorough knowledge of the subject". A few respondents, however, express diverging views at this stage, highlighting potentially opposing points of view that could be exploited during the training program. These views include placing value on student initiatives and errors (this idea was expressed in two excerpts), concern with taking students' difficulties into account (expressed in five excerpts), and concern with showing the relevance of mathematics (seven excerpts), with adapting one's teaching (11 excerpts), and with emphasizing comprehension over finding the right answer (three excerpts).

Changes in the Views Held by Student Teachers

An analysis of the answers given by the graduating student teachers (*cf.* Table 1) reveals major changes in their views of problem-solving activities. They no longer regard such activities as the simple application of computational rules, or as simply following a procedure (# 9 and 15). Also evident are significant changes in the importance they place on definitions (# 3), the use of symbols (# 8), and the language of mathematics (# 12)—factors that they no longer regard as representing the limits of mathematical activity. Consequently, they no longer place emphasis on learning formulas and algorithms (# 36) or on the use of symbols in all stages of teaching (# 55 and 63). Instead, greater emphasis is placed on materials, and on exploration and representational activities. Lastly, their view of the process of learning mathematics has also evolved. It is no longer regarded as mere imitation (# 21); children are perceived as coming to school with knowledge in hand (# 19), and errors are no longer viewed as the result of carelessness (# 18); the learning process is now regarded as a more complex activity in which the student plays a key role (# 23, 31) This new way of regarding the student is also evident in their answers to the open-ended question. The following types of statements are found: "Encourage the students to participate; create opportunities for them to succeed; the students must be at the centre of the discovery and learning process; challenge the students and provide a framework in which they are free to express their ideas; listen to the students, get them to participate so that you can see where they are having difficulty; involve them actively in the learning process; let the students discuss, think, reason with and challenge each other; to do so, create situations that are sufficiently enriched and likely to be of interest to the students. A good teacher must foresee and understand his or her students' errors in order to help them progress."

Conclusion

The few examples cited above are indicative of the changes that occur in the student teachers' views on mathematics teaching. The view of the teacher as a skillful communicator seeking to pass on his or her passion and knowledge gives way to that of a teacher who is more centred on his or her students, their reasoning processes and their errors, and who is seeking to initiate an interactive thought process involving activities carried out with the students. Further analysis is required to identify more precisely (beginning with the different entry profiles) the changes that take place in student teachers' views at the different stages of the training process and the aspects of the training strategies used that actually contribute to these changes.

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A STUDY OF PROPORTIONAL REASONING AMONG SEVENTH GRADE STUDENTS

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Contextual problems involving rational numbers and proportional reasoning in three broad categories -- rate, ratio and scaling -- were presented to seventh grade students with different curricular experiences. There is strong evidence that "new" curricula students who are encouraged to construct their own procedures for solving proportions and applying those skills collaboratively to applied problem solving, perform better than traditional curriculum students. In any case, seventh grade students are capable to recognize situations in which ratio or proportional comparisons are appropriate and do have the ability to represent ratio and proportion flexibility and in some cases even accurately.

Introduction

Recently new curriculum, new strategies and new emphases have been developed concerning many of the topics of middle school mathematics. This is especially true regarding the treatment of rational numbers, including fractions, decimals, percents, ratio, and proportional reasoning. In traditional middle school curricula, each arithmetic operation with each type of rational number is taught with a focus on developing student proficiency in well-defined computational algorithms that are then practiced to ensure speed and accuracy of execution. Only when that computational proficiency is attained will students be challenged to apply their computational skill to practical or fanciful "word problems".

On the other hand, in the Connected Mathematics Project (CMP is a new curriculum for grades six, seven and eight; created at Michigan State University) the approach to rational numbers and proportional reasoning is to encourage students to construct their own procedures for doing rational number computations, solving proportions, and applying those skills to applied problem solving. The CMP curriculum supports that construction of rational number knowledge by presenting students with a series of contextual problems requiring proportional reasoning and computation. Students collaborate in work on the problems, sharing their diverse insights and approaches with partners and then with the whole class through Mathematical Reflections discussions and journal writing. At no point in the CMP curriculum materials are students shown any standard algorithms for addition, subtraction, multiplication, or division of fractions or decimals. They are not shown standard procedures for solving problems involving percents (e.g. the "three cases of percent"), nor any routine method for solving proportions or testing ratios for equivalence (e.g. "cross-multiplication").

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The striking difference between traditional and CMP approaches to rational number and proportional computation and problem solving raises a very natural and fundamental question by parents, and others, who are concerned about the performance of students at this level. The question is: How do the computational and problem solving strategies and success of CMP and traditional curriculum students compare?

In particular, it is natural to wonder whether the new CMP approach does successfully lead students to construct effective (accurate and/or efficient) strategies for fraction, decimal, percent and proportional computation and whether CMP students develop flexible and/or effective strategies for solving contextual problems involving rational numbers and proportions.

Purpose of Study

The basic goal of our proposed study is to describe the character and effectiveness of proportional reasoning by seventh grade students with different curricular experiences as they face problems in the following three broad categories (following ideas of Freudenthal, 1978; 1983):

- Comparing magnitudes of different quantities with an interesting connection, as in "miles per gallon", or "people per square kilometer", or "kilograms per cubic meter", or "unit price". These computations are not generally called ratios, but rates or densities.
- Comparing two parts of a single whole, as in the "ratio of girls to boys in a class is 15 to 10", or "a segment is divided in the golden ratio".
- Comparing magnitudes of two quantities that are conceptually related, but not naturally thought of as parts of a common whole, as in "the ratio of sides of two triangles is 2 to 1". These comparisons are sometimes referred to as scaling and they include questions of stretching and shrinking in similarity transformations.

The main purpose is to compare the two populations. However, we are also interested in learning more about how seventh grade students learn and what they know about proportional reasoning. For an extensive review of the literature on proportional reasoning see Tourniaire and Pulos (1985) and Behr, Harel, Post and Lesh (1992).

Methodology

CMP sites for testing were selected based on the criteria that students had studied two full years of CMP: the sixth grade and seventh grade. Five different sites were identified: Portland, Sturgis and Shepherd, MI, San-Diego, CA, Pittsburgh, PA. Control sites were selected from the overall population of control sites for the CMP assessment. Based on matching with the CMP sample sites, and on Iowa Test of Basic

Skills (ITBS) results, control classes were obtained from Parma, MI, Toledo, OH, San-Diego, CA, and Pittsburgh, PA. The CMP sample consists of eight seventh grade classes and the control sample consists of six seventh grade classes. In total, 187 students were in the CMP sample, and 128 students were in the control sample.

The instruments designed for this study consist of students written tasks, students structured interviews and teacher written questionnaire. Three major types of proportional reasoning problems were chosen to be included in the written Proportional Reasoning Test: rate problems, ratio problems, and scaling problems. Three forms were created. The rate problems were included in forms 1 and 3, the ratio problems in forms 1 and 2, and the scaling problems in forms 2 and 3. In each class each form was distributed evenly among the students as much as possible. The 4 rate problems were attempted by 124 CMP students and 91 control students. The 5 ratio problems were attempted by 124 CMP students and 85 control students. The 5 scaling problems were attempted by 126 CMP students and 80 control students. The tests were administered in May, 1995.

For example, the rate problems were grouped around a story about "A Trip To The Zoo": Max, Eliza, Alex, and Cosima planned a bicycle trip to the zoo as a year-end outing for their class. Students gathered at the school parking lot and rode together on the bicycle path to the zoo. After looking at the animals for a few hours, they met at the picnic tables near the duck pond for a snack and cold drink before riding back to school. [The first problem was:]

1. Max and Eliza bought supplies for snacks and reported the following expenses: Gatorade cost $2.00 for 16 ounces. Cran-raspberry juice cost $1.60 for 12 ounces. They bought Cran-raspberry juice. Did they make the most economical choice? Show the calculations that lead you to that answer.

The ratio problems were grouped around a story about "On The Road To School" and dealt with: different ways of comparing the numbers of groups of students; equivalent and inequivalent ratios/fractions; and missing value problems.

The scaling problems dealt with photos, enlargements, and shadow images. The students were asked to identify enlargement factors, deal with equivalent and inequivalent ratios and find area relationships.

A special rating form was created to analyze the data. Three major categories were identified: Correct Answer, Incorrect Answer, and Blank. Correct Answer has three sub-categories: "Only the correct answer", "Correct answer with correct support work", and "Correct answer with incorrect support work". These sub-categories were created because in each problem the students were asked to provide support work by providing reasons for their answers. The incorrect answer has also three sub-categories: "Only incorrect answer", "Correct thinking but wrong conclusion", and "Incorrect thinking". The analysis of the data included determining the percentages in the cells on the basis of the total number of students multiplied by the number of problems of that form. For example, the Rate form included 4 problems which were attempted by 124 students, so the total N of the distribution is 496. In this case, each entry is a percentage of that total.

Results

The overall results are presented in Table I. It can be seen , that most of the students responded to most of the problems with support work. If we exclude the "Only correct answer", "Only incorrect answer", and the "Blank", 75% of the CMP students and 66% of the control students provided support work. Nevertheless, the quality of writing is important. Our impressions are that the CMP students demonstrated more proficiency in this regard than did the traditional students.

Table I: Proportional Reasoning – Overall Results

CMP Students vs. Control Students (All numbers in this table are percents)

		RATE		RATIO		SCALING		OVERALL	
		CMP N=496	CNT N=364	CMP N=372	CNT N=255	CMP N=630	CNT N=400	CMP N=1498	CNT N=1019
CORRECT	Only the correct answer	2	5	5	1	4	4	3	4
	Correct support work	51	28	43	21	36	16	43	21
	Incorrect support work	7	14	9	20	1	4	5	12
INCORRECT	Only the incorrect answer	2	6	4	5	17	17	9	10
	Correct thinking, but wrong conclusion	13	10	6	5	2	2	7	6
	Incorrect Thinking	18	28	19	20	23	30	20	27
BLANK		7	9	14	25	17	27	13	20

Obviously, the correct answer with correct support work is the most desired response from students. The overall results show that the CMP students outperformed the control students (43% vs. 21%). Looking separately for each type of problem, both samples were better in rate problems and worst in scaling problems. In the Blank category, the percentages increase from rate, to ratio, to scaling. Overall, the CMP students' responses were blank 13% of the time, while 20% of the control students' responses were blank. Obviously, as the topic is more instructional-related, the percentage is higher. The analysis of the data includes many additional comparisons between specific items for both samples. For example, a special focus was on the application of additive vs. multiplicative principles. Another attempt was to identify students' strategies and methods in dealing with a certain type of proportional reasoning as required in the first "Rate" problem (given in the Methodology section). Ten different strategies (correct and erroneous) were identified, for the majority, we could identify students work to demonstrate the application of the strategies. For one or two strategies which might be too sophisticated for seventh grade students, we complete the picture by providing our own analysis.

The followings are the ten identified strategies with some students' examples for the first problem in the "Rate" form:

(1) Comparing the ratios of two different variables using "external ratios" or a "functional method" as mentioned by Tourniaire and Pulos (1985). Actually, it is the "unit rate" strategy dealing with "price per unit" or "unit per price". For example – student's work:

2.00÷16   12.5 Gatorade   12.5¢ unit price.
1.60÷12   13.3 Cran-raspberry   13.3¢ unit price.
No, they didn't make the best economical choice.

This answer was classified as Correct with "correct support work".

Another example –

2.00÷16 .125
1.60÷12 .133
Yes, they made the best economical choice.

This answer was classified as Incorrect within the sub-category of "correct thinking but wrong conclusion".

Another example of Student's work :

16 $2   1 13¢   12 $1.60 1 $.14
No, they didn't make the best choice.

This is an example of directly comparing ratios of two different variables.

Another example of using "unit per price":

16 $2   8   12 $1.60 7.5
No, they did not; because with Gatorade you get more for your money and with C. juice you get less.

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(2) Comparing ratios of the same variable using "internal ratios" or a "scalar methods" (Tourniaire and Pulos, 1985).

For example – incorrect answer of a student:

16 12 1 R4
200 1.60 1 R40
Yes, they made the right choice.

This answer was classified as "Correct thinking but wrong conclusion". We could not find many students who used this strategy.

Our completion is: 16 ounces/12 ounces = 1.333... = 4/3

⇒ the quantity on the numerator is a better buy.

2.00$ / 1.60$ = 1.250 = 5/4

(3) Comparing the cost of the same quantity by finding common factor or common multiple quantities – "price per unit" is a specific example for this strategy.

For example – student's work:

\$1.60 3 $.53\overline{3}$
+ .53
2.13 Gatorade \$2.00 for 16 ounces
Cran-raspberry \$2.13 for 16 ounces.
No, they didn't make the best choice.

Another example is: No, . . . for 48 ounces Gatorade is \$6 and C.R.B. is \$6.40.

(4) Comparing amounts for the same cost by finding common factor or common multiple costs – "unit per price" is a specific example for this strategy.

For example, one student used the "building up" strategy (the next one) and found that he/she can buy 60 oz of Cran-raspberry for \$8 and 64 oz of Gatorade for \$8.

Of course, one could compare for 40¢: \$2.00/5= 40¢ ⇒ 16/5 = 3.2 oz Gatorade

\$1.60/4= 40¢ ⇒ 12/4 = 3 oz C.R.

or find the amount per \$1: 16/2 = 8 oz per \$ Gatorade

12/1.60 = 7.5 oz per \$ C.R.

(5) "Building up" strategy by using a list or a table.

For example – student's work:

G		C	
$	ounces	$	ounces
2.00	16	1.60	12
	32	3.20	24
6.00	48		36
		6.40	48

No, they didn't because at 48 ounces of juice Gatorade costs \$6.00 even and Cran-raspberry cost = \$6.40.

(6) Looking at the ratio of the differences between the same variable.

For example – students' work:

2 00	16	
-1 60	-12	NO
40	4	

16	2 00	
-12	-1 60	YES
4	40¢	

Many others used more or less the same strategy, but we could not find an acceptable explanation.

The following is our mathematical explanation for the "difference method".

Price per unit	Quantity	Cost
type 1 → x	n of type 1	nx
type 2 → y	m of type 2	my

Assume $x>y \Rightarrow x = y + z$ Assume $n > m$

Using the difference method:

$$(nx-my)/(n-m) = [n(y+z)-my]/(n-m) =$$

$$(ny+nz-my)/(n-m) = [y(n-m)+nz]/(n-m) = y + nz/(n-m)$$

This is the "price" of one additional item of type 1. Obviously, it is more expensive than type 2.

(7) A strategy of relating to only one variable by ignoring part of the data in the problem. Obviously, this is erroneous strategy.

For example – students' work:

Yes, because each drink you buy is 40¢ cheaper, so you are saving lots of money.

No, because Gatorade is cheaper, because it has more ounces to it.

(8) A strategy of responding just to the numbers.

For example – students' work :

Yes
1.60x12 19.2
2.00x16 32

2.00x12 24 Gatorade
1.60x16 25.60 Cran-raspberry

(9) A strategy named by us as "affective responses". We identified two kinds: by the value of money (under or over) and by taste (like or dislike).

For example – students' work :

Yes, they did because it really didn't cost that much money.

No, Gatorade tastes better

No, because what if some of the kids don't like Cran-raspberry.

Yes, it was a good choice, they saved 40¢.

(10) The last strategy includes: method used is not clear, answers are given but no method is given, no response.

As mentioned before, we interviewed students from both samples and administered teacher's questionnaire. This was done in order to gain an additional insight to the written tasks, especially when students were also asked questions without stories such as "is the ratio 4/7 equal to 10/13?" or "is 7/8 = 8/9?".

Conclusion

While the CMP students outperformed the control students throughout the study, we can see that we are dealing with some very difficult ideas to master. The authors of Street Mathematics and School Mathematics, Nunes et al. (1993), say that they have found no linear path of learning through this complicated maze of proportional reasoning. Our findings confirm those thoughts. The variety of ways students find to solve problems is always amazing. Our task is to keep the doors to clever solutions open to those students who will produce them.

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Children's Word Meanings and the Development of Division Concepts

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Introduction

Focus

Many researchers agree that division concepts begin developing in students as early as age seven but are not clearly understood by most students until age 18. Work has been done to examine students' responses at various ages to a variety of division problem types, different division contexts, and rational number division (Kouba, 1989; Tirosh & Graeber, 1989; Harel, Behr, Post, & Lesh, 1994). Division strategies and division models have been studied by Fischbein, Deri, Nell and Marino (1985), Kouba (1989), Greer (1992), Confrey (1994) and others. For this study, we wanted to extend the investigation of children's division concepts beyond their processes, strategies, and models. More specifically, we were interested in children's meanings of words that are commonly used in division instruction.

Theoretical Framework

Discourse and negotiated meaning are current areas of pedagogical interest in the reform agenda. They emerged with the constructivist perspective of the mathematics classroom where teacher and students mediate the meanings of words, symbols, and situations (Vygotsky, 1986; Voight, 1994). For Vygotsky (1986) word meanings were essential links between language and thought. Evidence suggests that students use language differently to express their thoughts and before adolescence have very different meanings for words than adults (Vygotsky, 1986). For mathematics educators who ascribe to constructivism and promote the role of classroom discourse, it becomes important to examine the range of meanings that students have for words commonly associated with division.

Luria (1981), a student of Vygotsky's, used card sorts as a method of eliciting responses from children. The children's considerations of each word while sorting and classifying become an opportunity for them to verbalize their thoughts about the words and other word associations.

Voight (1994), and Lo (1994), among others, used case studies to examine the mathematical meaning of elementary children. From these studies, negotiating meaning within the social context of the classroom and the role of the teacher in sense making emerged as notable conditions for learning mathematics. These findings suggest that it is important for teachers to realize that even though students use the teachers' words, the students' meanings may be very different from those of their teachers. Berenson and Vidakovic (1995) observed that students in grades 3-8 have many different meanings for division words such as "sharing" or "fair share", reflecting the diversity of cultures represented in schools today.

Method

The 38 students in this study were in grades 5-8 in a large, rural county in Southeastern United States. They were selected based on the results of earlier structured interviews given by their teachers, which we thought were representative of the range and diversity of division ideas found among middle grades students. The interviews for this study were conducted by the researchers. Sources of data included individual interview transcripts, video tapes, field notes, and students' paper work.

We chose to adapt the methodologies of Luria (1981); the first method is concept definition and the second method is free classification. The basic idea of concept definition is to study students' definitions of division; it is accomplished by analyzing the associations of the word division that students make when defining the word. According to Luria (1981) there are several ways that the students may verbalize their definitions. In concrete associations, a student identifies some characteristic of the word division or relates the word to a concrete situation. In the second type, a student relates the word division to a category or system of concepts which Luria refers to as verbal-logical associations. For this study, students were asked to give their definitions of division after they had done the free classifications.

The method of free classification involved giving the students a number of cards, each with one word on them. The words used in this study were associated with division and were obtained from the students' teachers (See Table 1). The students were asked to group the words, name each group, and explain the reasons for placing the words in a particular group. We asked the students to perform three card sorts. After the first sort, the interviewer inquired if the student could decrease the number of groups by merging or regrouping the first word sort. These directions were repeated for a third sort.

Analysis and Results

Concept Definitions

The analysis of these data are incomplete at this time. A preliminary analysis found evidence that some students at this level do give concrete examples of division situations. For example, one sixth grade student described division as dividing candy among friends and another as putting an equal number of balls in boxes. Some students demonstrated difficulty in verbalizing their division definitions, picking out only one or two concept features such as "parts" or "groups." *Whenever you divide two numbers and it's like the people, like you have a party and you get a number for something for everybody, and you would divide it into – to be how many people you could get for it.* Still others were able to make verbal-logical associations with division. *Division is to take away from one big group ... just divide the big group into separate parts that are fair to each other.* What is less clear at this point in the analysis are the links between the students' concept definitions and their free associations.

Free Associations

Early in the interviews it became apparent that many students perceived

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two categories of words among the 16 they had been asked to sort. Students described some of the words as "math" words and others were described as used outside of the mathematics classroom or "non-math" words. For example, one eighth grader said that she had "*never heard of split, halve, and fair used in mathematics.*" A student in seventh grade described fair and share as things you have to do in life, whereas all the other words were math group words. Another student grouped split, separate, evenly, and equal as words associated with "marriage." There were some interesting examples of contradiction among the students. For example, one student claimed that she could not sort "split" into any group but then proceeded later in the interview to repeatedly use "split" in her descriptions of division. The dichotomy of terms perceived by the students prompted us to classify the card sort words as 1) division labels and 2) division descriptors. These are also shown in Table 1.

Table 1. Division Words and Researchers' Classifications for Card Sort Analysis

Division Labels	Division Descriptors		
remainder	split	share	equal
dividend	separate	total	same
divisor	share	fair	group
divide	parts	evenly	
quotient			

First sort The analyses of the word classifications began with an examination of the students' first set of word groups to determine which of the 16 words were associated with division. Nearly two-thirds of the students (n=24) named one of their groups "division, divide," or "dividing" in the first sort. Of these, there appeared to be three distinct types of word group associations. By far the most common division association (n = 18) was one that included only the labels of division: divide, dividend, divisor, quotient, and remainder. Two students associated several division descriptors such as halve, parts, group, and *split* with division, but did not include the division labels such as dividend and quotient. Four students demonstrated that they associated both division labels and descriptors with division within the first set of word groupings. For example, an eighth grader grouped *divide*, *divisor*, *dividend*, *quotient*, *remainder*, *split*, *equal*, *group*, *separate*, *halve*, *evenly*, *total*, and *quotient* and named the group "division."

Subsequent sorts. Fourteen students were able to associate all 16 words with division in the second or third sorts. Among these students were 3 who did not associate any of the words in their first sort with division. The remaining 11 students who were able to group all 16 words as "division" words had either used division labels or both division labels and descriptors in their first grouping. The logical progression of most students' associations seemed to begin with division labels in the first grouping, the addition of some division descriptors with the labels in the second grouping, and finally the inclusion of all the words as division words in the third grouping. When we considered the ages of these 14 students, there is little evidence that older students associated more words with division than younger students. For example, 3 fifth graders, 3 sixth graders, 6 seventh graders and 2 eighth graders were among these 14 students with multiple meanings for the concept of division.

Discussion

Further analysis and results will be reported at the PME meeting this summer. We will determine if students tended to associate each word with the name of the group or the group words with one another. Additionally, the many different names created by the students to name their groups are of interest.

There appeared to be a logical progression of word associations among these middle grade students. The first series of associations began with the associations of division labels, then the inclusion of words such as *separate* and *split* , termed division descriptors. Several possible avenues are open for future investigations including teaching experiments and classroom observations. These

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settings can provide students with opportunities to negotiate the meanings of words commonly used in division instruction.

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THE USE OF EXAMPLES IN THE TEACHING AND LEARNING OF MATHEMATICS

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In this paper I make a brief survey of literature on the use of examples in concept formation and compare abstraction from examples with concept formation from single or generic examples. I apply these theoretical ideas to a number of classroom incidents and conclude that there is a role for multiple examples even where the generalisation is generic.

Classical theories of the psychology of learning take as one of their foundations the human ability to distinguish, to identify sameness and difference, like and unlike, and thereby to group, separate and classify. The notion of classification allows us to conceive of members of a class and hence of representatives of classes, or examples.

The issue of classification is also a route into consideration of particular and general. "Particular" describes features of an individual member of a class, whilst "general" describes features common to all members. A statement describing an attribute of a particular member of a class might be adapted to describe an analogous "general" feature of every member of the class.

Many writers have considered the issue of particular and general in mathematics. Traditional theories of concept formation by abstraction are one attempt to characterise the relationship between particular and general in the learning of mathematics. Several major theorists have queried classical ideas about the nature of the relationship between general and particular in the learning process.

Classical theory has us forming concepts by abstraction of the commonalties from numerous encounters with the particular. Skemp (1971) uses the example of a child developing the schema of "chair" through numerous encounters with examples and non-examples of chairs. Dienes (1960) bases his principles for the teaching of mathematics on the notion of abstraction from examples.

Borasi (1984) expresses disquiet at this interpretation of learning new concepts in the context of mathematics education. She points to psychological evidence produced by Tall and Vinner (1981) which contradicts the notion that "irrelevant" attributes of the examples from which a concept has been abstracted will be forgotten once the concept is established. They found, on the contrary, that some features of the examples which were presented in the teaching of the concept were not attributes of the concept. Nevertheless they were retained as part of the students' "concept image", that is the students' mental picture of the meaning of the concept. Borasi also expounds the shortcomings of the abstraction model in the case of the concept of an infinite set.

Freudenthal (1978) argues that learning of mathematical concepts at school does not take place by the process of abstraction:

> "the origin of general ideas, concepts, judgements and attitudes in the learning process, whether they are attained in a continuous process, by *comprehension*, that is by generalising from numerous examples, as is the common opinion, or by *apprehension*, that is by grasping directly the general situation, which is my thesis." (p170)

He describes methods of promoting "apprehension" in the classroom by the use of "paradigms", that is single examples which give access to the general situation.

Lakoff (1987) argues a similar case concerning the classical view of categorisation. He opposes to it a new theory of categorisation called "prototype theory" and claims that "...prototype theory....suggests that human categorisation is essentially a matter of both human experience and imagination - of perception, motor activity, and culture on the one hand, and of metaphor, metonymy, and mental imagery on the other" (p8). Further, prototype theory suggests that there are good and bad examples of members of a category. This contradicts the classical view that no example of a category is any better than any other example and, says the author, fits better with our experience.

Other writers have expanded on traditional understandings of abstraction without stressing their limitations. Dreyfus, for example (1991) speaks of abstraction as focusing on relationships between objects rather than on the objects themselves. This description includes the traditional idea of shifting attention to the similarities and differences between objects, but also expands on it.

Harel (1991) treats abstraction as part of the process of generalisation and concept building. This part of the approach seems problematic in the case of the function, where there is evidence of students using all kinds of erroneous schema which they have abstracted from the examples that have been presented to them. He suggests use of generic examples as a means of assisting students in making abstractions and building concepts around formal definitions. I will refer to this work again in my discussion of generic examples.

In rejecting the classical abstraction model of concept formation, Freudenthal (op cit.) states his preference for a teaching method which employs "paradigms". A paradigm he describes as "one example, which evokes the general idea" (p170) or the one necessary example. In the context of learning Latin "amo" as an example of a first conjugation verb is a paradigm. It acts as a paradigm even though the transposing to other first conjugation verbs may be unconscious.

The notion of an example which is seen in some way as representing a generality has been taken on by a number of authors, often using the term "generic example".

Mason and Pimm (1984) discuss generic examples in a variety of contexts, suggesting $f(x) = |x|$ as a generic example of a continuous but non-differentiable function, 2/3 as generic example of the set $\{2t/3t : t \in Z\}$, and Kleenex as a generic example of a tissue. They point out that the role of example is to help students to see the generality which is represented by the particular. In other words students need to see the examples as "examples of" some more general statement.

In Mason (1993) Mason again points out that the teacher's experience of "examplehood" when presenting an example to students may be quite different from the students' experience.

Hazzan (1994) and MacHale (1980) draw attention to some of the dangers of over-reliance on canonical or generic examples. Hazzan made a study of students' understanding of group theory and in particular their ability to solve the equation $x = x^{-1}$ in the context of a group. Many of the students claimed that the only possible solution to this equation was $x = e$. One of the author's suggested explanations for this is that the students are relying on multiplication on the real numbers as their canonical example of a group operation, so that they assume that the only element which is self-inverse is the identity element. Features of the example which are not a part of the generality it represents, have been imputed to that generality. Hazzan links this over-use of the canonical example with the role of metaphor in understanding abstract concepts. The students see the group operation *as* multiplication, rather than *like* multiplication, so that one student says "Suddenly, everything (in Abstract Algebra) looks so strange. I mean why isn't $a*b$ equal to $b*a$?" (p53). These findings illustrate some of the points made by Tall and Vinner (op cit.) in their work on "concept image".

MacHale regrets the fact that text book authors are so consistent in their counter-examples, so that, for instance, $f(x) = |x|$ is almost the only example to be found of a continuous but non-differentiable function. The use of a single counter-example supports "monster-barring" (Lakatos 1976), that is it allows students to dismiss the counter-example and maintain their belief that, for example, all continuous functions are differentiable. In addition it does not encourage students to locate what it is that is similar about these examples and that makes them representative of the general. This amounts to an argument against the use of generic examples.

Harel (op. cit.) emphasises the generic example as a means of generalisation for students. He speaks of "generic abstraction" as the process of forming a new concept by consideration of one paradigmatic or canonical example and suggests three principles for selecting effective generic examples:

The *entification* principle says that the context from which the new object's properties are to be abstracted must be familiar. The *necessity* principle states that students must be able to see the reason for the abstraction they are being asked to make. The *parallel* principle says that the generic example must be treated in a way which can be paralleled later in the general case.

His last principle perhaps misses the point that it is the student's treatment of the example which is crucial. "Irrelevant" properties of the example may continue to form part of the student's concept image.

Balacheff (1988) uses the notion of generic example in the context of students writing proofs. Of his four categories of proof, generic example is the third and is characterised as follows: "The generic example involves making explicit the reasons for the truth of an assertion by means of operation or transformations on an object that is not there in its own right, but as a characteristic representative of the class" (p219) He suggests that such a "proof" is a step on the way to the formal "thought experiment".

The generic example then is seen as a stage between particular and general. It has been advocated as a teaching approach and observed as a stage in understanding. To see generic understanding as a stepping stone between particular and general is to deny the universality of the "abstraction from particulars" model of concept formation.

This discussion of the role of examples in the formation of concepts alerted me to look for the teachers' and students' use of examples in mathematics. Are there instances of the use of generic examples, and if so in what context? Is there evidence of students using examples as a basis for abstraction? These, and many other questions were in my mind as I undertook a period of teaching in a local school and also a series of meetings with a group of teachers. The classes I taught and observed were of seventeen year old students. During this time I made notes on incidents which struck me as relevant to my interests and also tape recorded some lessons and conversations.

During one of my meetings with the group of teachers we discussed a recording I had made of my conversation with a student. He was working on finding the equation of a straight line. Towards the end of this discussion, one of the teachers, Kate said:

> "This has actually just shed some light on a conversation I had with my son. He was finding equations of straight lines through a point and I was saying to him "use $y-y_1 = m(x-x_1)$"and he said "I've never heard of that before" and he wrote down for me $y = mx + mx_1 - y_1$ and I said "where did you get that?" sorry "$- mx_1 + y_1$" I'm getting it the wrong way round myself, and I said I had never seen it in that form before and he said "Well I did a lot of examples and I found that this pattern was working out" and it's the first time I've ever heard - I hadn't realised what I was hearing at the time - it's the first time I've ever heard of somebody coming up with their own generalisation from doing a lot of numerical examples - and I now think my son's quite clever actually"

Although Kate expresses surprise at her son's generalisation I can recall similar occasions from my own teaching experience. For example, two boys, Paul and Kwok, working on the Cartesian equations of circles arrived at a general condition that the equation $x^2 + y^2 + ax + by + c = 0$ should represent a circle. They did this whilst working through an exercise which contained a large number of particular questions of the same type.

It is important to note that Kate's son apparently came to his generalisation by spotting patterns in numbers. He saw a relationship between the coefficients in the equations he derived and the gradients and co-ordinates of points that were given in the questions. Paul and Kwok did not do this but worked through, in the general case, the procedure they had been practising in several particular cases. They performed on the general equation $x^2 + y^2 + ax + by + c = 0$ the process which they had routinized on numerical examples.

What the two incidents have in common is that the students came to an algebraic expression of their own generalisation in the course of working on a lot of particular cases. There is however a subtle difference between the ways in which they arrived at these generalisations. I might describe the former as an empirical or inductive generalisation and the latter as a generic generalisation. I find the labels "abstraction from examples" and "generic abstraction", as I have understood them in the work of others and described them in the earlier part of this paper, useful in making this distinction.

Having made this distinction I want to consider two further incidents in which I found these labels useful. The first is a conversation between me and one student, Ewan.

I ask Ewan to work on the question

> "Find the equation of a straight line which has gradient M and passes through the point (p, q)".

He says "Let's try it with $y = mx + c$" and writes this down but then doesn't have a strategy for starting. He claims that he could do the question if he had values for M, p and q so I ask him to work with M = 4, p = 2 and q = 3.

He draws a sketch of the line in this case but then says he has forgotten the method for finding the equation. I take him through the steps of substituting known values into $y = mx + c$. We don't write anything more down but Ewan works out a value for c in his head, saying

> "Yes. It's 8 plus something equals 3. .... 8, it would be 8 minus 5. Yes. So that's got to be -5. So it's got to be $y=4x-5$."

Next I ask him to work on the original question:

(In this transcript a series of full stops indicates a pause of half a second for each full stop)

1 Liz: Uhmhm. Right, now the job that you've been given

2 Ewan: Uhmhm.
3 Liz: is to find the equation of a line
4 Ewan: yes
5 Liz: which doesn't have them specified as numbers.
6 Ewan: Yes. So that, $q=mp-c$. *(writes $q=mq$-c)*
7 Liz: No do you mean p there?
8 Ewan: I do mean p, not q. .. p. I can't write either.
9 Liz: .... What did you use that equation for when you were doing the other one?
10 Ewan: That?
11 Liz: Hmm.
12 Ewan: I used .. p is x because it's the x -co-ordinate
13 Liz: Uhmhm.
14 Ewan: q is y, because it's the y co-ordinate. *(writes $q=mp+c$)*
15 Liz: Hmm.
16 Ewan: m is the gradient and c is the constant. And because I didn't know the constant but because I knew the other ones
17 Liz: right
18 Ewan: I knew that $mp+c$ had to equal q, so I could just work out what c was.
19 Liz: Right. Well the same is true for this case.
20 Ewan: Yes. ...... So it would be ..... $q=mp$. *(writes $q=(mp)$)* .... a bit of a shot in the dark ... 5 is what the two co-ordinates were when added together.

My intention in this interchange was that my example of the equation of a straight line going through (2, 3) and with gradient 4 should be a generic example for Ewan. I expected him to grasp the method and be able to apply it in the general case. His speeches in lines 12 to 18 indicate that he had grasped the method at some level. However, he does not go on, as expected, to manipulate the equation $q=mp+c$ to give an expression for c. Rather he goes back to the numerical example we had done to look for a number pattern. Perhaps his hesitation over doing this ("a bit of a shot in the dark") was because he had only one example from which to generalise.

I could speculate on the reasons for Ewan's failure to do the algebraic manipulation, but that is not my purpose here. I want merely to suggest that empirical and generic generalisation are confusingly (to both teacher and student) mixed together in this incident.

A common reaction from teachers with whom I have discussed this account is to suggest that I should have done more numerical examples of a similar kind with Ewan before asking him to work on the general case. This suggestion runs counter to the idea of my numerical example as a generic example. The essence of the generic example is that only one is required.

The second incident is from my meetings with teachers:

Three teachers were working on the following problem:

> In how many ways can *n* 1 by 2 rectangles be arranged to form a 2 by *n* rectangle?

Two teachers working together and one working on his own had independently come to the conclusion that the sequence of numbers of arrangements for increasing values of *n* was a sequence of Fibonacci numbers. Prompted to try to justify this conclusion, David, who was working alone, showed me how to obtain all the arrangements of four rectangles by adding two more rectangles to the each of the arrangements of two, and one more rectangle to each of the arrangements of three. I asked him to show his demonstration to the other two.

David: If it's Fibonacci, for number four you add the two combinations and three combinations together

Valerie: Right

David: There are my two combinations, three combinations so I just need to add one to each of those and I need to add two to these which if I add that way round I end up with all five combinations. .... hmm? So ..

Katherine: Why does - ?

David: They're the twos

Katherine: What happens if you add to the other side? Is it not possible to get any different ones?

David: I think that's going to be exactly the same results as if I'd added them on top. As long as I put these ones across and these ones down

Katherine: Because those two are the other way round - yes

David: Now - I haven't tried, but I guess three and four - I'm just assuming at the moment that it's just adding on - so that's four - and threes were - one, two, three (*laughter as David "secretly" takes some more rods from the two women's work*) so I should be able to get all the combinations just going like that, that, that

I had asked David to give his demonstration because I thought it would serve as a generic proof that each term was the sum of the previous two. In fact both David and the two women seemed to want to look at another case, that of $n=5$, in order to be convinced.

I suggest that these teachers were not looking for empirical evidence that their conjecture was correct. They had already seen that the sequence of numbers was a Fibonacci sequence. They were looking for confirmation of an argument, not of a result. In other words, they were looking through the particular to the general, rather than seeking statistical evidence.

On the basis of my brief review of literature on examples I might distinguish between empirical and generic generalisation on the basis of the number of examples needed. That is, empirical generalisation requires a number of examples whereas generic generalisation requires only one example. However, Paul and Kwok made what I would identify as a generic generalisation on the basis of a large number of examples (it is possible that they could have done as much after only one example) whilst Ewan attempted an empirical generalisation on the basis of only one example. The three teachers felt the need to look at a second example even though they were using a generic argument. The number of examples used is not a reliable indicator of the type of generalisation.

My study of examples in use suggests that a multiplicity of examples may be useful even in cases where their interpretation is generic. The distinction between the two kinds of generalisation may not be so easily made in practice as in theory.

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THE INFLUENCES OF SIGNIFICANT OTHERS ON STUDENT ATTITUDES TO MATHEMATICS LEARNING.

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Abstract

In the first part of a three year study in Australian secondary schools, Year 7 and 9 students were surveyed and interviewed about their attitudes towards mathematics learning. Factors investigated were their perceived performance at mathematics and the influence of peers, teacher, parental aspirations and experience of mathematics. Teachers and the students' parents views were also sought, and in this way apparent conflicts could be explored. In this paper we present certain quantitative findings and extracts from an interview which provide insight into the apparent influences of significant others described in the questionnaire data.

1. Background and research procedure

This study is generally concerned with understanding the reasons for the underdevelopment of the mathematical potential of many young Australians. Its focus is on students who may be experiencing cultural conflicts in terms of gender, ethnicity or class, and who may not be able to achieve their full potential in mathematics. The conceptual context for the research is student attitudes and in this paper we report on the first part of the study, which was carried out in 1994 with students in four predominantly Anglo-cultural background secondary schools, and which looks predominantly at gender differences.

Explanations and interpretations of under achievement in mathematics have tended in recent years to move away from the cognitive domain and to focus more on student attitudes (McLeod, 1992). Moreover as the social dimension (Bishop, 1985) has come to be recognised as a highly significant factor in mathematics education, so the need has arisen to carry out studies which examine the roles of particular individuals and groups in influencing young people's attitudes towards mathematics and mathematics learning. There is therefore the need, as Leder (1992) affirms, to adopt 'research paradigms that allow a greater attention to individual differences and context-specific problems.' In this study we are exploring the network of perceptions and relationships involving the individual students, their peers, their mathematics teacher and their parents. These are assumed to be the most 'significant others' (Sullivan, 1955) likely to influence the individuals' attitudes.

The four state co-educational schools were selected on their low non-English-speaking-background student numbers (less than 30%) , and on the socio-economic status level in the school's catchment area, with two schools having medium to high levels and two medium to low. In each school a Year 7 and a Year 9 class were chosen through consultation with the mathematics coordinators and the teachers concerned. Each teacher was asked to identify four low and four high achieving mathematics students. Four sources of data were collected: questionnaires administered to all the students in each of the classes studied, video tapes of three of their mathematics lessons, interviews with the selected students, and interviews with their parents (wherever possible). Full ethical procedures were followed.

2. Questionnaires.

2 (a) Questionnaire development

To determine the students' attitudes towards various aspects of mathematics learning, a multi-dimensional questionnaire was developed, using items from the Fennema-Sherman Attitude Scales (1976), the Mathematics Attribution Scale (Fennema, Wolleat & Pedro, 1979), the Individualised Classroom Environment Questionnaire (Fraser, 1990) and items developed by the researchers based on statements from The National Statement on Mathematics (Australian Education Council, 1991).

There were four sections to the questionnaire. ***Your Views about Mathematics*** included items to ascertain each student's attitudes to mathematics and to learning it. ***More Views about Mathematics*** assessed the students' attributions for their success or failure in mathematics in terms of ability, effort, task and environment.. The ***Individualised Classroom Environment*** questionnaire was used to ascertain each student's own perception of the learning environment within their mathematics classroom. ***How Good Are You?*** (Forgasz & Leder 1995) aimed to determine the students' perceptions of their own ability and how they thought their parents, teacher, and peers would rate them as learners of mathematics.

2 (b) Questionnaire Results

Our general results are consistent with earlier findings of students' perceptions about mathematics. Girls indicated they were more anxious about mathematics and felt more strongly than boys that mathematics was not a male domain. Furthermore, low achievers perceived they obtained less support from their teacher and during interviews they said it was hard to get attention, or felt ignored by their teacher, or even avoided the teacher. The reasons students gave for avoiding the teacher included being confused by their explanations, they were shy, they thought they would be bothering the teacher, or were afraid of revealing a lack of understanding.

In the selected questionnaire results to which we refer here, students were asked to rate their own achievement level in mathematics on a scale from 1 (low) to 5 (high), which we discuss below as High Achievers (ratings 4 and 5) Middle Achievers (rating 3) and Low Achievers (1 and 2). Each student also gave their 'wished for' rating, their perceived teacher's, classmates', mother's and father's ratings of their achievement, as well as their perceived mother's and father's 'wished for' ratings for them.

Students' self-rating compared with their 'wished for' rating

Students overwhelmingly indicated they wanted to do better at mathematics (84%). Fifteen percent of students were happy with their level of performance. There appears to be a gender difference in this latter result as 21% of boys stated they were satisfied with their performance while only 6% of the girls did so.

Students' self-rating compared with teacher rating

Perhaps the most 'significant other' which should first be considered is the student's own teacher. When the students' self-ratings were compared with their teachers' ratings, based on end-of year results, over half of the students overestimated their performance. Boys were also more likely to overestimate and girls more likely to underestimate their performance ($p<0.01$, Table 1).

Willis (1990) reported that boys over-rated and girls under-rated their performance in mathematical achievement in relation to written assignments. Under-rating of performance has been assumed to be associated with girls having lower self-esteem than boys. Yet it is not clear from the literature whether over-rating of performance is more productive, that is whether it contributes to higher achievement levels, than under-rating of performance. Our data indicate that this is not necessarily the case since the teachers gave the girls a higher rating than the boys, although the difference was not significant (one-third of a mean grade point difference out of 5 grades).

Table 1: Students' self rating of performance by gender.

	Over-rating of achievement	Self rating and achievement level same	Under-rating of achievement
Girls (n = 79)	44%	29%	26%
Boys (n = 96)	67%	23%	10%
Total (n=175)	56%	26%	18%

Moreover, when the same data were analysed in terms of the year of school attended, Year 9 students also over-rated their performance to a greater extent than the Year 7 students (p<.01), while the teachers rated the Year 7 students significantly higher than the Year 9 students (p<.001). Perhaps under-rating of one's performance is related to factors other than self-esteem? We now turn to our data on the influence of other 'significant others'.

Students' self-rating compared with perceived class-mate rating

Table 2: Peer influence in the over-rating and under-rating of performance in mathematics.

	'Peers over-rate my performance'	Peer and self rating the same	'Peers under-rate my performance'
Girls (n = 80)	28%	62%	10%
Boys (n = 99)	15%	58%	27%
Total (n=179)	21%	60%	19%

Within the classroom the students' classmates are likely to be a highly significant influence. In relation to their perceived rating from their peers, the boys were more likely to believe that their peers under-rated their achievement ($p<.01$) while the girls were more likely to believe that their peers over-rated their performance (Table 2). It is important to note, however, that the majority of both girls and boys perceived no difference between self and peer rating.

A question on the attribution scale in the questionnaire **More views about mathematics**, also made reference to the impact of distracting peers as a factor determining failure. A highly significant interaction was detected between gender and student perceived achievement for this question ($p<.01$, Table 3). An examination of the means showed that while distracting peers diminished as a factor in perceived failure for higher achieving boys, for the higher achieving girls it increased.

Table 3. The influence of distracting peers as a factor in failure
"Imagine you have not been able to keep up with the rest of the class in maths this term: Students sitting near you wouldn't work."
Scale 1 - 5. 1 = strongly disagree up to 5 = strongly agree (mean and sd in parenthesis)

perceived rating	boys (98)	girls (81)
5 excellent	2.4 (1.2)	3.8 (1.0)
4 very good	2.8 (1.1)	3.3 (1.1)
3 average	3.4 (1.2)	2.7 (1.1)
2 below average	2.6 (0.9)	3.0 (1.3)
1 weak	3.5 (1.4)	3.4 (1.3)

Similarly, in relation to year levels, the influence of distracting peers in general was greater in Year 9 (mean 3.2; sd 1.1) compared to Year 7 (mean 2.8; sd 1.1; $p<.01$). This result suggests peers are a more significant factor affecting performance in the mathematics classroom in Year 9.

Student self-rating compared with perceived parental ratings

The third group of significant others whom we considered were the students' parents. In relation to their perceived parental ratings, students overwhelmingly perceived that their parents wished them to do better than their current performance (78%). Only nineteen percent of students perceived their parents were satisfied or at least did not wish for a better performance in mathematics.

More interesting, although more difficult to interpret, are the findings related to the student's mothers and fathers seperately. For example, boys perceived that their mothers would rate them higher than did the girls ($p<.05$) and the same pattern was observed for the perceived father rating although the difference was not significant. In addition, students who rated themselves as low achievers believed that their mothers would rate them higher at mathematics than their fathers would. This suggests that mothers and fathers exert rather different influences on their children's performance. However, when the teachers' rating of the students' actual performance was used as the independent variable, this finding was not replicated.

Our conclusion is that students' perceptions of others' ratings of their mathematical performance interact with their own in different ways, and exert complex but important influences on their attitudes, and therefore on their behaviours in the classroom.

3. Student Interviews

Fifty-students were selected for interview but space limits prevent reporting these in detail here. (Further reports on the interview data will be provided at the conference.) We include below selections from one interview with a female Year 9 student, Kerry, to illustrate some of the effects of the influences of significant others on her behaviour in the classroom.

Kerry was designated as a very successful student in mathematics by her teacher; in fact in her view she topped the class. Kerry, however, under-rated her own performance level, saying she was "a bit above average". She predominantly works in the class with a middle achiever, Beth:

Interviewer: Is it important who you sit next to in class?
Kerry: Doesn't affect me at all, except when friends ask for help all the time.
Interviewer: Who would they be?
Kerry: Like Steven and Beth.
Interviewer: They want to look at your work?
Kerry: Yes.
Interviewer: Is it hard to not respond to that?

> *Kerry: Yes, sometimes, because if I don't help Beth she gets annoyed with me. Like they ask you in the middle of a question when you are trying to finish your question and it is really hard.*
> *Interviewer: If you really wanted to excel in class, who would you sit next to?*
> *Kerry: Not like to sit next to, but probably if fairly good people sat together it would be better.*
> *Interviewer: Do you sit with people who are good at mathematics?*
> *Kerry: No, not really, sometimes I try and sit with people who are fairly good.*

While Kerry was aware that she might be more comfortable working with someone equal to herself in mathematics, she felt unable to alter the social dynamics of the classroom.

> *Kerry: I enjoy doing maths, though not something you tell people, you get looked at funny. In primary school I used to say it was my favourite subject!*
> *Interviewer: What sort of people would look at you funny? Both boys and girls?*
> *Kerry: Both but mainly the boys. Mainly boys who don't do so well. I can tell you the girl, Beth, she gave me a hard time about a lot of things.*

Kerry is clearly aware of the conflict that Beth creates for her by simultaneously asking for help while denying Kerry the opportunity to develop her own abilities. Kerry revealed further the damaging influence of peers when asked to identify a good and bad learning experience in mathematics.

> *Kerry: You could look at this one both ways. Getting 100% on a test, everyone lays it on you. (They) call you a square. That is really hard. I try and ignore it, keep it to myself but it wears on you, sometimes I go off at people.*
> *Interviewer: Is the same pressure on boys who achieve well?*
> *Kerry: Not as bad. Normally it is the opposite sex and girls wouldn't go off so much as boys.*

In the literature, girls are portrayed as more supportive of each other and less competitive than boys. For example, girls are reported to place a greater priority on friendships, which signifies "an expression of a cultural emphasis on solidarity" (Wyn, 1990, p.125). Yet what strongly emerges from Kerry's comments and from her actions in the mathematics classroom is evidence of enormous peer pressure, particularly from her female work partner, for Kerry to camouflage her positive attitude to mathematics. Under-rating her performance serves to be functional for her in order to remain socially acceptable while still managing to succeed.

The influence of girls upon other girls' attitudes towards mathematics is clearly one area in need of further exploration as the questionnaire results did provide evidence of less peer pressure on high achieving boys compared to girls. More generally, the impact of peers within the social arena of the classroom undoubtedly contributes to student anxiety in mathematics and points to the need for teaching which focusses upon the development of functional group dynamics in the mathematics classroom.

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STUDENT TEACHERS' APPROACHES TO INVESTIGATIVE MATHEMATICS: ITERATIVE ENGAGEMENT OR DISJOINTED MECHANISMS?

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This paper reports on the first phase of a project which aims to characterise student teachers' approaches to a range of problems of an investigative nature with the aim of identifying and analysing the different influences that affect these strategies. Results are discussed for a group of 14 first year primary school trainee teachers, with more detailed analysis being presented for 5 case study students. The study has been influenced by research on problem solving and didactique of mathematics. The main findings suggest that more successful students interact with their written text in an iterative as opposed to a linear way. Tabular representations are sometimes used mechanistically and unproductively as separators of activity. Interviews with students played a crucial role in identifying their strategies, which were not always evident from the written script alone.

Introduction and Background

In the 1980s problem solving became one of the major themes in Mathematics Education. As Schoenfeld (1992) points out, despite the statement from the National Council of Teachers of Mathematics that "problem solving must be the focus of school mathematics" (NCTM, 1980 p1), there is a very broad view as to what problems and problem solving actually involves. This ranges from applying standard techniques in routine exercises, to thinking creatively about some situation (often not explicitly mathematical). Exploring any patterns involved, posing problems and seeking solutions using formulation, testing and proof of conjectures are also aspects of problem solving. This type of mathematical activity finds its roots in Polya's problem-solving strategies (Polya, 1957) and has been developed in work on mathematical thinking by, for example Mason et al (1982). In various parts of the world official or national publications promoted changes to the curriculum, suggesting a move towards "mathematics as an exploratory, dynamic, evolving discipline" in the US (National Research Council, 1989) and open problem-solving and investigations in the UK (Cockcroft Report 1982). The subsequent changes in the UK led to problem-solving and investigations becoming part of the official mathematics curriculum. The inclusion of such activities as assessed coursework for GCSE (examination at age 16) has ensured their being common occurrences in UK schools. Similar emphasis on the study of mathematics as a science of patterns has influenced Australian curricula (McGregor and Stacey 1993, Australian Educational Council 1991).

There has recently been a debate about the effectiveness of such mathematical investigations in the UK. Hewitt (1992) has questioned whether the diversity and richness of such open ended problems is being reduced to spotting patterns from tables. Wells (1993) in a contentious pamphlet introduces the notion of Data-Pattern-Generalisation (DPG) as a general mechanistic method of solving problems said to have little or very limited mathematical value. Relating this to Teacher Education he suggests that in some courses "students are expected to do investigations...thus, being inducted into false and very limited ideas of mathematics" (p48). The overall thrust of these arguments is that the potential positive advance of pupils exploring mathematics at their own level has been seriously undermined by the algorithmic and mechanical nature of the approaches adopted by pupils in schools. Barnard & Saunders (1994) also maintain that an instrumental understanding of content is being replaced by an instrumental understanding of process.

Much of this research has not considered the dynamics of the teacher-student relationships within an institutionalised educational setting. Some of the more negative effects of the use of open problem solving could be explained by Brousseau's (1986) notion of the metacognitive shift in which perceived failure on the part of students can lead to the teacher imposing heuristics as objects of study instead of the mathematics intended. Brousseau suggests that this phenomenon is more likely when heuristics, advice and models are given the status of cultural objects and he uses Venn diagrams in the "modern mathematics" movement as an example of this effect (Brousseau 1986). This phenomenon is not due to inadequacies on the part of teachers and pupils, but is in fact an inevitable (or at least potential) consequence of any teaching situation. The institutionalisation of meta-level guidance on how to approach open problem-solving can readily be seen in UK curriculum materials. For example in materials produced by the Shell Centre (1984, p.46) the following key strategies are recommended: Try some simple cases; Find a helpful diagram; Organise systematically; Make a table; Spot patterns; Use the patterns; Find a General Rule; Explain why it works; Check regularly.

Cox and Brna (1995) maintain that external representations are used by those students who are successful within a problem solving situation to monitor their performance and provide a source of explanation. It is also suggested that graphical representations are more limited in terms of expressing abstractions than sentential representations (Stenning and Oberlander, in Press) and that because of this they may provide more vivid self-explanation feedback than a more linguistic modality such as language and algebra. Cox and Brna also report that students who were successful in solving problems testing analytical and verbal reasoning were more likely to have used multiple representations. They also found large differences in the types of external representations used by these students and they attribute one source of this variation to cognitive style.

Given this background it was decided to carry out a research study to characterise student teachers' approaches to a range of problems of an investigative nature, with the aim of identifying and analysing the different influences that affect these strategies (Blanc 1995). This paper presents the results from the first phase of this research.

The study

The setting for this phase of the study was an initial teacher education course at King Alfred's College, Winchester, working with first year students. These students, who are training to become primary teachers with a specialism in mathematics (14 in the group, 11 females, 3 males), follow a four year programme leading to an educational degree with qualified teacher status. This cohort included mature students and recent school leavers with a variety of both educational and social background. Students who

have recently left school are likely to have very different experiences of pre-18 school mathematics than mature students because of recent changes in UK curricula.
The students worked on a range of investigative problems throughout the year, incorporated into their normal teacher education programme. In this paper we report their work on the diagonals of a polygon problem (Fig 1). The rationale behind the choice of this problem was that, whilst straightforward, it can be solved by a variety of strategies, it was suitable for the group as a starting problem and was related to the field of graphs and networks which was the content area for this session. There is also previously reported research (Balacheff 1988) of younger students' attempts at this problem, where some analysis of their problem solving is given.

How many diagonals does a polygon have?
Investigate.

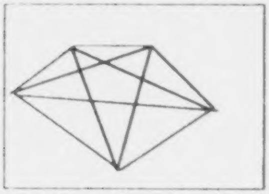
Produce a written solution to this problem
Make sure that you explain your solution
Ensure you don't erase any work
you reject. I'm interested in all your work

Fig 1

Data were collected by: analysis of student scripts (photocopied); taping of the session (dictaphone); notes made by the observer during the session and immediately afterwards. All students completed a questionnaire on their educational and mathematical background. Five students (offering a range of background and experience) were selected for more detailed micro-analysis of their scripts and interviewed in depth (taped and transcribed) to further probe their solution strategies. For readers unfamiliar with this problem, one way of expressing the solution is that for a polygon of n sides there are: $n(n-3)/2$ diagonals.

Two possible routes to this solution are:

Method 1: Let n be the number of vertices of the polygon ($n \geq 3$). There is no diagonal from the vertex to itself or to its two neighbours so the number of diagonals leaving each vertex is $n-3$. The total number of diagonals is therefore equal to the number of vertices multiplied by the number of diagonals leaving each vertex. This must be divided by two since each diagonal gets counted twice.



Method 2: A table of numbers of vertices and diagonals is as follows:

No of Vertices	No of Diagonals
3	0
4	2
5	5
6	9
7	14

a) We notice that the differences between the number of diagonals go up as follows 2,3,4,5. The next case will be a difference of 6. So for 8 vertices there are 14+6=20 diagonals.
b) For 7 sides there are 5+4+3+2 diagonals. This can be recognised as the sum of the natural numbers up to 5 with 1 subtracted. So for n sides the number of diagonals is the sum of the natural numbers up to n-2, subtract 1. This is $((n-1)(n-2)/2)-1$.

Analysis of students' strategies

Table 1 presents an overview of the characteristics of the students' approaches to the "Diagonals of a Polygon" problem Space does not permit inclusion of the full micro-analysis of each student's approach, this will be covered more fully in future articles Students' strategies were characterised according to three main criteria:

- **types of external representation used** (annotated diagrams, tabular representations and algebra and natural language); reflecting the distinction between graphical/diagrammatic and propositional/sentential representations, discussed in Cox & Brna (1995), tables being an intermediate representation
- **the nature of students' solutions**, this includes whether students used their method of counting, visual cues, or the table directly in their development of a rule, characterised as recursive or universal, expressed in algebra and/or in words We also indicate whether a satisfactory general solution of any kind is generated
- **ways of working with text**, some students' moved around the script going backwards and forwards (iterative) whereas others used a linear approach Evidence of switching backwards and forwards between specific representations is also noted

Table 1

	Anne	***Beth***	***Len***	***Wendy***	***Sue***
(D) Annotated Diagrams Used	Many, with repeats	Many with repeats	Many no repeats	Many no repeats Multiple annotations	Many no repeats Originals enhanced later
(T) Tabular Representations Used	Minimal information Differences[1] Divides activity	Minimal information Differences Divides activity	Two tables First has Differences Divides activity Second similar	Two tables First numbers Second with algebraic ratios of vertices to diagonals	Three columns Includes diagonals per point Differences
(A) Algebra and Natural language used	Use of Arithmetic Prog (AP) Variables and constants confused	Minimal algebra Different meanings in different contexts	Extensive with manipulation Only after tables	Extensive with manipulation and use of summing AP	Rules expressed in various forms Algebra as end product
Development of a rule.	Recursive from AP formula	Recursive from table	1) Recursive from table 2) From algebra	Recursive from counting/ drawing strategy	Universal from visual cues in diagrams/table
Representation and form of final solution.	No final solution	Partial solution only, algebraic (Recursive Rule)	1) Words (Universal Rule)	Algebra (Universal Rule)	Words /Algebra (Universal Rule)
Satisfactory general solution	No	No	Yes	Yes	Yes
Switching between representations	Minimal None after use of table	Minimal None after use of table	Switching between T and A	Extensive Flexible between D, T, and A	Extensive Enhancement of D and T leads to refined A
Working:	Linear	Linear	Mixed	Iterative	Iterative

We now present an overview of one student's work, *Sue* Superficially her script reads as work done in a strict linear way, perhaps following mechanistic recipes, but when interviewed it was clear that she jumped about on her script, in an iterative way adapting her representations to suit her needs When *Sue* reached an impasse she enriches her representations, adding information to diagrams and the table She developed her first general rule from the table and the diagrams This solution was then further refined using a relationship derived from the tabulated information. Whilst the table is used to find patterns this student's work is more than just mechanistic pattern spotting The table is used as organisation and as a focus for activity going both forwards, further into the problem as well as back re-examining work done before

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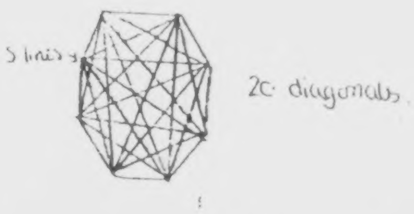
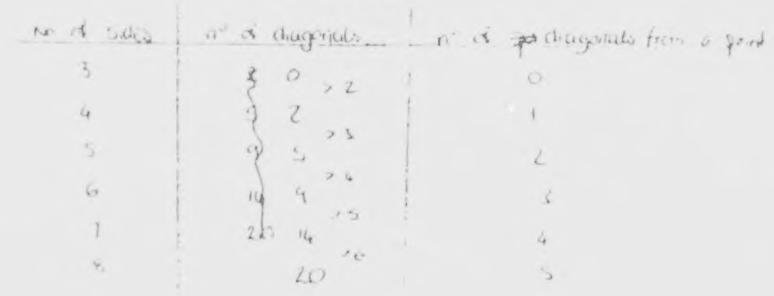
[1] This indicates students calculated the differences between the numbers of diagonals and wrote this on their script

Sue draws examples for the triangle, quadrilateral (a square), pentagon, hexagon, heptagon and octagon writing the number of diagonals on the right Written on the left of each shape is the number of lines from each point This was **not** done when drawing out the shapes and diagonals originally but considerably later, after she had constructed her table

An example diagram



Sue's table



no of sides	n° of diagonals			n° of ~~po~~ diagonals from a point
3	3	0	> 2	0
4	5	2	> 3	1
5	9	5	> 4	2
6	14	9	> 5	3
7	20	14	> 6	4
8		20		5

The list of (first) differences (2,3,4,5,6) added to the second column appears to lead to a hypothesis which predicts the next case from the previous cases In the interview *Sue* said that she drew the table because she wanted to look at the numbers, to see the results together The third column of the table comes directly from the diagrams but was added later *Sue* simply counted up how many lines came from each point, that is made use of the visual image *Sue* said she thought about how she drew the shapes in order to get to the solution "I was drawing all the diagonals from one point to go to the next one ..." so her generation of the rule comes, at least in part, from drawing in the diagonals

$$\frac{(\text{n° of lines from each point} \times \text{n° of sides})}{2} = \text{n° of lines}$$

It is important to note that both the table and the diagrams underwent several transformations. *Sue* stated that she still needed to draw the shape out to find the number of diagonals from each point. She then wrote that by subtracting three from the number of points you get the number of lines from each point and provided a refined formula in words. She wrote " you don't have to draw it out"

$$\frac{n^\circ \text{ of sides x (no of sides - 3)}}{2} = n^\circ \text{ of } \cancel{\text{lines}} \text{ diagonals}$$

When questioned where this insight came from she said "that would have been from the table". *Sue* appears to negotiate with herself by going back over work correcting, refining and enhancing previous representations.

External Representations

The main external representational devices used by the students were annotated diagrams, tabular collection of information, algebra and natural language. We found that it was not whether a particular device was employed that mattered but how this device fitted in to the overall solution which differentiated the outcomes.

Cox and Brna (1995) say that in impasse situations students frequently switch representations arguing that time spent in constructing new representations is a heavy burden. We would add that exactly how this switching (routinary, self-directed or 'thrashing') takes place is important. They also suggest that **graphical representations** can serve to illustrate structure. The way students make use of these graphical representations from the point of view of mathematics appears to be related to how they have constructed them (and whether they have attended to this in their solution). For example *Wendy*'s recursive solution comes directly from her own constructions (see Table 1). She states: "Count all diagonals from one vertex then move on to the next vertex...each time we changed vertex the number of lines to be drawn decreased, this made us think of a series formula." The counting/drawing method here is a crucial element of solution. This suggests that constructing diagrams for yourself as opposed to being presented with a static constructed diagram could make a difference to the problem solving process. Of the students in this small sample (Table 1) one saw a visual solution (2 out of the cohort of 14). This seems to be in contrast with the results of Balacheff (1988) in which more students seemed to make use of the visual structure. Learning geometry could promote a visual awareness. Does the UK students' lack of geometry explain their not paying more attention to the visual figurative image?

The most striking phenomenon of **tabular representations** was confirmed at interview. A tabular representation was used in an inflexible way by 3 of the 5 students studied in depth. Two of these stated that they had received strong advice about doing tables and the third said she had been strongly influenced by her partner to draw a table. In these three cases the table seems to act as a **separator** so that work after the table uses only the table itself as a potential source of information. This supports the contention of decontextualised pattern spotting (Wells 1993). However tables were used in a highly flexible, dynamic way by *Sue* (as illustrated above) and as a means of organisation by *Wendy* (she was adamant that the tables were to sort out "the muddle" not to spot patterns, her rules are derived from counting/drawing methods). There is weak evidence that once students begin to use **algebra** they do not switch from this external representation and this issue is one focus in the ongoing research.

We have evidence that some students do use different representations as strict divisions of activity and that this may be due to inflexible use of some taught process model of problem solving. Identifying such behaviour is far from straightforward as we have shown in the case of *Sue*. If students did see the work as separate sections with their own beginnings and ends this may interrupt their solution process. We are particularly interested in whether this separates the generation of the rule from the generation of the data they use to derive the rule. We note that of the students who were most successful in solving the problem, their engagement was the overriding concern. External representations were only useful to them in so far as they helped them continue to solve the problem, less successful students seemed to use the representations as mechanisms.

Linear and Iterative working

Individual interviews with the students, after they had completed the problem highlight the very different ways of working with the written text. Some students work in a linear fashion down the page with little reworking or looking back. This led in some cases to discontinuities in solution and failure to exploit potentially crucial information. *Beth* (see Table 1) recognises similarity to the handshakes problem and uses this method to demonstrate a recursive counting strategy (counting down from (n-1) to 1) Yet after drawing the table, she does not exploit this solution in her work. *Beth* and *Anne* both stated (categorically) that they did not look back after the table in contrast to *Wendy* and *Sue* whose attention moves all over the work, that is, they use their written text in a non-linear way using a varied range of representations in an **iterative** manner adjusting, correcting and enhancing. This is not usually evident from an analysis of the written script. In fact it is likely that mechanistic approaches may be inferred from a surface analysis alone.

Some Concluding Remarks

Open ended problem solving has been introduced into schools as a reaction against the teaching of routinary, algorithmic methods. Almost inevitably, new mechanistic (DPG) methods are being incorporated into the institutionalised practice of such investigative work as Brousseau's research indicates would be the case. This is why Brousseau suggests that it is more important to emphasise the devolution of a problem as opposed to the type of problem being solved.

This study has shown that there are considerable differences between the ways in which students interact with paper-based external representations when solving open mathematical problems. In agreement with the work of others (Lesh et al. 1987) the most successful students, from the point of view of solving the "Diagonals of a Polygon" problem (cf. Table 1), interacted with external representations in an iterative way, often using several external representations in parallel (switching), returning within the process of solving the problem to modify or extend an external representation as *Sue* did.

Is it possible to teach students to pay attention to mathematical structure when constructing diagrams? If so it will be inevitable, as shown by Brousseau, that any form of teaching could institutionalise this practice as a new heuristic which could replace the mathematics being taught. This didactical phenomenon does not imply that we should stop teaching but that all forms of teaching bring with them potentially negative effects. It also suggests that teacher training should emphasise such phenomena.

Subsequent phases of this project have involved analysing students' work in computer-rich situations, for example using spreadsheets or Cabri to solve open problems. We conjecture that these environments are likely to structure the students approaches in particular ways, for example spreadsheets could provoke students to construct tables and graphs. Work with Cabri may provoke students to focus on more graphical information. The nature and extent of the ways in which students make use of external representations developed within computer-based activity when working away from the computer will also be the focus of ongoing research.

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THE ANALGEBRAIC MODE OF THINKING AND OTHER ERRORS IN WORD PROBLEM SOLVING

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Abstract: *A conceptual framework is given for examining students' solutions to complex algebraic word problems. A characterization of errors is given, distinguishing errors related to analgebraic mode of thinking from errors related to problem analysis and to management difficulties. new forms of analgebraic errors are revealed.*

1. Introduction

Many studies have been concerned with issues of general problem solving as they apply to algebraic word problems. One of these issues is to what extent problem solvers use semantic knowledge and other domain related knowledge in their solutions. Another issue is solvers' ability to recognize the structure of a problem and to apply a relevant solution schema. Studies concerning these issues are surveyed by Chaiklin (1989).

An issue which is particular to algebraic word problem solving is that of translation processes. Most studies which investigate this issue deal with simple problems of the "students and professors" type, and with the reversal error. An extensive survey of these studies, as well as an account of their own contribution was given by MacGregor and Stacey (1993). Cortes (1995) classifies errors in the translation of word problems.

In this paper I would like to focus again on both general and particular. As to the general, I will use Schoenfeld's characterization of mathematical problem solving performance (1985). As to the particular process of translation, I will focus on the analgebraic mode of thinking introduced in Bloedy-Vinner (1995), and bring new instances of it as well.

2. The study

Contrary to most studies which deal with the translation process, this study examines students' solutions to complex algebraic word problems, namely, problems which state several relations of varied types. The problems chosen for the study were standard text book type problems. Although the mathematics education community does not always approve of this type of problems, these are the problems which are currently being used by the system for teaching translation skills, and can therefore reveal translation difficulties.

The purpose of this study was: 1. To characterize errors caused by algebraic language difficulties. 2. To distinguish those from errors caused by other factors. As will be shown, the investigation of complex problems gives us the opportunity to examine the factor of algebraic language versus the other factors, and to reveal new misconceptions of algebraic language, which are not revealed in simple problems.

The problems were given to Israeli students who had taken matriculation exams before, and were studying at a university preparatory course. By a rough estimate, more than half of high school graduates are on their mathematics level or below. Some of the students were interviewed after solving the problems.

3. Theoretical framework

Schoenfeld (1985) suggests four categories of knowledge and behavior which serve to explain mathematical problem solving. These categories are: Resources, heuristics, control, and belief systems. Two of these, resources (knowledge possessed by the individual that can be brought to bear on the problem at hand), and control (global decisions regarding the selection and implementation of resources) are relevant to my analysis of skill components needed for solving standard algebraic word problems. These components are:

1. Ability to understand and analyze the relations stated in the problem.
2. Knowledge of algebraic language.
3. Management of unknowns and equations.

The first two components are resources, the third is control.

Understanding and analyzing the relations depend on semantic knowledge of words in the statement of the problem, on knowledge of mathematical concepts and facts (like geometric shapes and formulae, for example), and on pragmatic knowledge related to problem domain. Some of the relations are not stated explicitly, and must be gathered by pragmatic knowledge. Illustrations will be given in the following sections.

As to the second component, I am going to use the analysis of algebraic language and of analgebraic mode of thinking introduced in Bloedy-Vinner (1995). It was argued there, that algebraic language is poorer than natural language in noun types (numbers only) and in predicates ($=$ and $<$ only). This leads to difficulties in translating natural language predicates or relations which do not exist in algebraic language. Students may resolve these difficulties by erroneously enriching "their" algebraic language. This behavior is the **analgebraic** mode of thinking, namely, usage of algebraic language which does not comply with its standard mathematical meaning. Various forms of analgebraic thinking which were revealed in this study will be described later.

Management of unknowns and equations is the control needed when complex word problems are translated. It is the act of deciding how to manage the translation: which unknown numbers in the problem statement will be designated a letter, and which will be expressed by an expression constructed to translate a relation; whether to translate a relation by an equation or by an expression which is going to be used in the translation of subsequent relations. As a result of this management we end up with a number of equations and a number of unknowns which are determined by the problem.

The number of relations stated by standard word problems is usually such that we end up with an equal number of equations and unknowns.

Errors in word problem solving can be caused by failure in any of the components described here. In the following sections I will analyze students' errors and classify them according to these components.

4. Analysis of student's solutions

The students were asked to write equations which translate the problems, but not to solve them. Let me start with the following problem:

Problem 1. *Before the game Tal had 3 times as many marbles as Gadi. During the game Tal lost half of his marbles to Gadi, and then the number of marbles Gadi had was 12 more than the number of marbles Tal had.*

The relations stated (explicitly or implicitly) by the problem are:

(a) At the beginning Tal had 3 times as many marbles as Gadi.
(b) Tal lost half of his marbles.
(c) Gadi won half of Tal's marbles.
(d) At the end Gadi had 12 marbles more than Tal.

The variety of the 269 solutions analyzed was enormous, and this made the analysis very difficult. It was decided, therefore, to analyze the solutions relation by relation, classifying errors in the translation of each relation according to the first two components: understanding and analysis of the relation, and knowledge of algebraic language. In addition, the management of the solution was classified. The same method will be used here to describe the classification of solutions:

Translations of relations (a) and (d):

Starting the translation with relation (a), it could be translated by writing a two-variable equation, e.g. $X=3Y$, or by writing a one-variable expression, e.g. $3Y$ for Tal's number of marbles. On the other hand, ending the translation with relation (d), it could only be translated by writing an equation, equating expressions constructed (correctly or incorrectly) for the numbers after the game, adding 12 to Tal's number.

Let us first look at errors of analysis. Understanding and analysis of these relations require semantic knowledge of " times as many as" and " more than" and the distinction between them[1]. Confusing addition with multiplication in these relations was considered an analysis error. 7 students did that in relation (d) and none in relation (a) (see Table 1).

All other errors were classified as analgebraic. 31 out of the 32 analgebraic errors in (a) and 97 out of 115 in (d) involved reversals. The reversal error was dealt with in many studies, usually in two-variable equations translating simple one-relation problems. Most explanations, excluding MacGregor and Stacey (1993), were based on the interpretation of letters as objects, word abbreviations or labels. An additional explanation was proposed by Bloedy-Vinner (1995). According to that explanation, the origin and the image of an algebraic expression (which we understand to be a function) are vaguely conceived as one entity, being changed by the function, but remaining the same entity. For example, X and 3X are the same entity (e.g. number of Tal's marbles), becoming 3 times larger by the action of the function. This leads to the interpretation of 3X as the predicate "X is 3 times as large", a vague one-place predicate, not paying attention to the question larger than what. Interpreting expressions in the sense of predicates instead of functions, is an error which helps students translate predicates which do not exist in algebraic language, thus enriching "their" algebraic language. Algebraic expressions can, thus, tell stories, not just construct numbers.

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[1] The Hebrew statements of these two relations have the same syntax and the same words except for their prepositions.

Table 1: Distribution of translations for each relation in Problem 1.
(Analysis errors and analgebraic translations are not mutually exclusive.)

n=269	Correct translation	Analysis error	Analgebraic translation	Relation missing	No answer
Relation (a)	231 85.9%	0 0.0%	32 11.9%	1 0.4%	5 1.9%
Relation (b)	225 83.6%	7 2.6%	22 8.2%	10 3.7%	5 1.9%
Relation (c)	106 39.4%	118 43.9%	19 7.1%	27 10.0%	5 1.9%
Relation (d)	138 51.3%	7 2.6%	115 42.8%	6 2.2%	5 1.9%

Since we are dealing with a complex problem, the students had to use whatever they had constructed for relation (a) in their translations of the subsequent relations (b) and (c). This enables us to get evidence which supports the claim about analgebraic translation which was made above. Let us look at some examples (see Figure 1):

In Example 1 the student starts with declaring who is who, and writes a reversed equation for (a). Later, in the division 3Y/2, we can see that he considers 3Y to be Tal's number (he lost half his marbles). Thus, as his solution evolves, he identifies the origin Y and the image 3Y, both to be the same entity of Tal's number of marbles. This may be related to his interpreting his first equation as a table with two unequal numbers on both sides: 3Y - Tal's number, and X - Gadi's number of marbles, so that 3Y tells the story "Tal has 3 times as many marbles."

In Example 2 the student uses the children's initials, so we know who is who. He starts with a nonreversed equation, but then, in 2T/3 (probably a fraction error, instead of (T/3)/2) we can see that he considers T/3 to be Tal's number. Here, though the first and second equations are correct, the student may have read his second equation in a table-like manner, namely, Tal's and Gadi's unequal number of marbles on both sides, and thus T/3 became Tal's number of marbles. So again, the origin T and the image T/3 are the same entity.

Figure 1. Examples of solutions of Problem 1.

1. X- Gadi, Y- Tal,	6. TalX 3XGadi, 12 + X 2 3X
3Y X, 3Y 2 X + 12.	7. Tal 3X, Gadi X,
2. T 3G, T 3 G, 12 2T 3 G	Tal after game T T + 12Gadi after game
3. Tal- 3X, Gadi- Y, 3X 2 + Y 12.	X + 3X T T + 12
4. 3T G 1.5T + G + 12	8. Tal X, Gadi 3Y,
5. Before after	Before the game: 3Y X
Tal 3X 3X 2	After the game: 3Y-12 X
Gadi X X + 12 3X 2 X 12	9. Tal 3X, Gadi X
	10 3X 2 3X 2 + 12

In Example 3 the student chose to denote Tal's number by 3X, where X does not denote Gadi's number. In fact, X does not denote anything, and 3X is chosen just to tell the story "Tal's number is 3 times as large", interpreted as a vague one-place predicate. Example 4 is even stronger evidence for this: the use of the letter T implies that both T and 3T denote Tal's number, and 3T is used in the equation in the sense of "Tal's number which is 3 times as large".

21 out of the 32 analgebraic translations of (a) included direct evidence of the kind shown in Examples 1-4.

In translations of (d), 20 out of the 97 reversals included direct evidence that both X and X+12 are conceived as the same entity, Gadi's number, which is "12 more" (the question "12 more than what?" remains obscure). Example 4 is one illustration of that: The student writes a table-like equation, with total number of marbles before and after the game on both sides. we can see that both G on the left and G+12 on the right denote Gadi's number. Solution 5 is another example of that: X and X+12 are Gadi's number before and after the game. The table-like equation has Tal's and Gadi's numbers on both sides. In the interviews these students were asked whether they thought that Gadi had won 12 marbles, and they all said that they didn't think so, but rather that as a result of the game, his number became 12 more than Tal's.

We have seen that the analgebraic translation we are dealing with is related to the problem of "who is who". This difficulty is less likely to arise when translating (a) into the one-variable expression: There it is obvious that X is Gadi's number, 3X is Tal's number, and 3X translates correctly the relation between them. In this study students were not told to write a two-variable equation. 205 students chose to translate (a) into a one-variable expression, only 2.9% of which were reversed. 58 students wrote equations, 43.1% of which were reversed. This explains why in (a) there were few

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reversals as compared to (d) and to other studies reported in literature, where students had to write equations.

The examples we have discussed involved table-like equations with unequal entities of the children on both sides, like $3Y=X$ or $3X/2=X+12$. In addition to these, there was another type of table-like equations translating (a) and (d). The equality sign was used as a separator between an origin and its constructed image, like $X=3X$, $Y=Y+12$, or $3X/2=3X/2+12$ (see Examples 6, 7, 10.) 3 answers in (a) and 16 in (d) included this form of analgebraic translation.

Translations of relations (b) and (c):

Understanding and analysis of these relations require pragmatic knowledge of games where marbles pass from one player to the other, entailing that whatever Tal lost was won by Gadi (this was hinted but not stated explicitly). Also, mathematical knowledge of fractions is needed. Since relation (c) is not stated explicitly, many students weren't aware of it, and didn't translate Gadi's winning. Examples 1-7 illustrate this error, made by 109 students. As mentioned before, when asked about it in the interviews, they said that Gadi did win. Still, they weren't aware of it while they were writing their translations. 3 students thought that since Tal's number was divided by 2, Gadi's number should be multiplied by 2. The analysis errors in (b) included confusion about who had lost and what he had lost (5 students), and fraction errors like the one mentioned in Example 2 (2 students).

A new form of analgebraic error was revealed in 19 solutions: a letter or an expression denote the number of marbles of a child, and is considered to change as the story evolves, without performing any algebraic operations on them. The same letters and expressions are used to translate both (a) and (d), while (b) and (c) are considered to "happen automatically", maybe by writing the words "before" and "after" besides the equations. Example 8 illustrates this. In an interview about an age problem not reported here a student said about similar equations he had written: "*The first equation is true now, and the second equation will be true in 10 years*". As if the expressions have a life of their own and they change with time. I will call this phenomenon expressions or variables with evolving meaning .

Each of the relations was classified as missing (see Table 1) when the student either gave an uncompleted solution as in Example 9, or when he introduced a new letter without writing another equation to translate that relation, as in Example 7 (where (b) is missing).

As to the management, it was considered incorrect when there were too many equations (Example 6) or too few equations (Examples 3, 4, 9). It may be said that management errors are the result of analysis errors and analgebraic errors we have discussed.

The summary of solutions for Problem 1 is given in Table 2. It shows the contribution which each skill component made to errors.

Table 2: Distribution of Problem 1 solutions by error types they include (n=269).

Correct answer	Analysis errors	Analgebraic errors	missing relations	Incorrect management	No answer
76 28.3%	123 45.7%	146 54.3%	37 13.8%	37 13.8%	5 1.9%

I would like, now, to present the second problem. Because of lack of space, I will describe only aspects which are different from what we have seen in Problem 1. These aspects are characteristic of the geometric domain.

Problem 2: *When each side of a square was increased by 2 cm., it's area became 12 cm² larger.*

Figure 1. Examples of solutions of Problem 2.

1. $2X\ \ Y,\ 2(X\cdot 2)\ \ Y\cdot 12$	5. $(4X)2\ \ 12$	9. $(a\cdot 2)(b\ \ 2)\ \ S\cdot 12$
2. $2(X\ \ 2)\ \ 12$	6. $(X\cdot 2)4\ \ S\ \ 12$	10. $(a\cdot 2)b\ \ S\cdot 12,\ ab\ \ S$
3. $4X\ \ Y,\ 4(X\cdot 2)\ \ Y\cdot 12$	7. $2a\cdot 2b\ \ 12\cdot ab$	11. $x^2\ \ S,\ 12\cdot x(x\cdot 2)\ \ S$
4. $S\ \ 2X,\ (2X)2\ \ 12$	8. $(a\cdot 2)(b\ \ 2)\ \ 12$	12. $S_2\ \ a^2-12$

Understanding and analysis of this problem requires knowledge of the geometric domain. There were two types of errors related to this. First, 28 students used erroneous area formulae (see Examples 1-7). Second, 18 students misunderstood the problem, and thought it was about a rectangle (Examples 7-10), or about a square where only two sides were increased (Example 11). Other errors, which were made by 17 students and were also classified as analysis (knowledge) errors, included wrong order of operations (like writing $(X+12)^2$ instead of X^2+12, or X^2+2^2 instead of $(X+2)^2$).

The missing relation category includes 19 (13.6%) solutions which introduced a new letter for the area without writing a corresponding equation (e.g. $A=X^2$). In Problem 1 this only occurred in 1.1% of the solutions. 16 of these students had used the initial S of the Hebrew word for 'area'. They probably thought that using this letter, which was normally used in area formulae, already told the story of the area relation, and felt free, therefore, to use it without adding an equation. Examples 6, 9, 12 demonstrate this management error.

Some solutions may be interpreted as story like translations, where symbols are used like words. For example, (4X)2=12 may be telling the story: "4 sides were increased by 2 and the result is an increase by 12". This is another instance of erroneous enrichment of algebraic language mentioned earlier. These are not necessarily word by word translations, and may be telling the contents of the problem as understood after its analysis.

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The summary of solutions for Problem 2 is given in Table 3. It shows the contribution which each skill component made to errors.

Table 3: Distribution of Problem 2 solutions by error types they include (n=140).

Correct answer	Analysis errors	Analgebraic errors	missing relations	Incorrect management	No answer
46 32.9%	59 42.1%	11 7.9%	49 35.0%	44 31.4%	10 7.1%

5. Conclusion

We have seen an analysis and a classification of the solutions of two complex word problems. A method of analyzing each relation separately, and then integrating the results was used. This method made it possible to classify hundreds of different solutions of a problem. The complexity of the problems enabled us to get evidence of several forms of analgebraic errors: vague conception of an origin and an image as one entity, an expression interpreted as a predicate, table-like equations, variables or expressions with evolving meaning, and story like translations.

The errors were attributed to failure in 3 skill components: 1. Analysis of the problem and domain related knowledge, 2. Knowledge of algebraic language, 3. Management of the solution. The extent to which each component contributed to errors was investigated.

The problems which were presented belonged to different domains and included explicit and implicit relations. Because of that, they could illustrate the influence of the mathematical knowledge (geometry, fractions, order of operations), and the pragmatic knowledge needed to understand and analyze the relations in a problem.

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CHALLENGING THE TRADITIONAL SCHOOL APPROACH TO THEOREMS: A HYPOTHESIS ABOUT THE COGNITIVE UNITY OF THEOREMS

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The purpose of this report is that of highlighting the possibility that in an adequate educational context the majority of grade VIII students successfully implement a process of theorem (conjecture and proof) production, characterised by a strong cognitive link between conjecture production and proof construction. A detailed description is given of this process and of how it surfaced in a teaching experiment organized by us. The conditions are discussed that may have allowed the extensive implementation of the process in the classroom. Some educational implications are sketched.

1. Introduction

The purpose of this report is the introduction and justification (on an experimental basis) of a hypothesis concerning mental processes underlying the production of statements and proofs by VIII grade students.

The hypothesis stems from previous research on the feasibility of a constructive approach to theorems by students. In particular, during a teaching experiment concerning arithmetic theorems students were engaged in the production and proof of conjectures. It was observed that students kept a keen coherence between the text of the statement produced by them and the proof constructed to justify it (see Garuti & al., 1995). This textual coherence brought forward the problem of a possible cognitive continuity between the statement production process and the proving process.

The hypothesis forming the subject matter of this report is that *the majority of grade VIII students can produce theorems (conjectures and proofs) if they are placed in a condition so as to implement a process with the following characteristics:*

- *during the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingling with the justification of the plausibility of his/her choices;*
- *during the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ("arguments") produced during the construction of the statement according to a logical chain.*

Despite the undeniable differences between *"deductive organization of thinking"* and *"argumentative organization of thinking"* (Duval, 1991), we want to stress some aspects of continuity, concerning the production, during the construction of the conjecture, of the elements ("arguments") that are used later during the construction of the proof.

The hypothesis featuring as subject matter of this report, which concerns the holistic character of the theorem production, if validated and thoroughly investigated by other studies, might have important didactic consequences as to the school approach to theorems, radically calling into question the teaching traditions (see Discussion).

2. References to history and research in mathematics education

The history of mathematics shows remarkable similarities between the holistic way of producing theorems by the student, described in our hypothesis and the way of producing theorems by mathematicians: despite important differences (as to reasoning, cultural experience, institutional bonds, etc. - see Hanna & Jahnke, 1993), we can detect the existence of common features, in particular as to the intermingling between the progressive focusing of the statement and the argumentative activity aimed at justifying its plausibility. At times, in the case of the history of mathematics, this is a long process, that involves many people for many years (cf Lakatos, 1976); at times it is a personal process, traces of which are found in the notes or memories of one mathematician (cf Alibert & Thomas, 1991).

With reference to the theoretical approach to "hypotheses" proposed in Boero & al. (1995), the production of a conjecture as described by us in the Introduction can be considered as a "hypothesis" production act: that is to say, it can consist of the argumented selection (prompted by a given question made by the student himself or by others) among possible alternatives, with a margin of uncertainty, as to its validity, that can be solved through the systematically organised reasoning or a counterexample ("verification" of the "hypothesis").

In the research produced in Maher (1995), in a problem solving situation implying the necessity of formulating and justifying conjectures, a behaviour similar to the one described in this report is observed in very young students (grade IV).

All these elements prompted us to examine all over again the studies on the mathematical proof within the mathematics education research, which, on the contrary, above all point out the elements of difference between argumentative reasoning and deductive reasoning (Balacheff, 1988; De Villiers, 1991; Duval,1991; Hanna & Jahnke, 1993; Moore, 1993; Tall, 1995). It seems to us that the existence of differences, epistemological obstacles, etc. is not incompatible with the fact that students can construct the proof using elements come up during the argumentation that accompained the conjecture construction process. But every element of continuity implies the risk for students to identify processes of different nature (cf Duval, 1991). These reflections were helpful to us for the planning of our teaching experiment and for the analysis of students' behaviours: in particular:

- at the stage of construction of the teaching experiment we tried to create favourable conditions for the appearance of the cognitive unity assumed by us, but also for the spacing out by students of the conjecture production stage from the proving stage, insisting in particular on the reasons for the necessity of proof as "proof of the statement truth";
- in the analysis of protocols we tried to catch the signs of attained change in students between the perspective of the argumentation to construct the conjecture and the persuasion of its plausibility, and the perspective of its proof.

3. Description of the teaching experiment

The main difficulty which we had to face was that of finding experimental confirmation for our hypothesis. It was necessary, in particular, to create an experimentation and observation context suitable to "reveal" the nature of processes of statements and proofs production and verify the potentiality conjectured by us. Underlining indicates some crucial points.

The teaching experiment was carried out in two grade VIII classes of 20 and 16 students, at the beginning of the third school year with the same teacher. Students had already interiorized the habit of producing argumented hypotheses in different domains (mathematical and non-mathematical), writing down their reasoning. Students had already experienced situations of statements production in arithmetic and geometry; they had approached proof production in the arithmetic field (see Boero & Garuti, 1994; Garuti & al., 1995).

The task concerning the production and proof of a conjecture was contextualized in the "field of experience"(Boero & al, 1995) of sunshadows. Students had already performed about 80 hours of classroom work in this field of experience. They had observed and carefully recorded the sunshadows phenomenon over the year (in different days) and over the morning of some days. They had approached geometrical modeling of sunshadows and solved problems concerning the height of inaccessible objects through their sunshadows.

The field of experience of sunshadows was chosen because it offers the possibility of producing, in open problem solving situations, conjectures which are meaningful from a space geometry point of view, not easy to be proved and without the possibility of substituting proof with the realization of drawings.

In the two classes the activities were organised according to the following stages (whole amount of time for classroom work: about 10 hours):

a) Setting the problem :

"In the past years we observed that the shadows of two vertical sticks on the horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and an oblique stick? Can shadows be parallel? At times? When? Always? Never? Formulate your conjecture as a general statement."

(Individual work or work in pairs, as chosen by the students)

Some thin, long sticks and three polystyrene platforms were handed, in order to support the dynamic exploration process of the problem situation.

b) Producing conjectures: many students started to work with the thin sticks or with pencils. They started to move the sticks or to move themselves to see what happened. Other students closed their eyes. The absence of sunlight or spotlight in the classroom hindered the experimental verification of conjectures they were formulating: it was the mind's eyes that were "looking". Students individually wrote down their conjectures.

c) Discussing conjectures: the conjectures were discussed, with the help of the teacher, until statements of correct conjectures were collectively obtained which reflected the different approaches to the problem by the students.

d) Arranging statements: through different discussions, under the guidance of the teacher, the following statements,"cleaned" from metaphors and more precise from a linguistic point of view than those produced by students at the beginning, were collectively attained:

-*"If sun rays belong to the vertical plane of the oblique stick, shadows are parallel."*

- *"If the oblique stick moves along a vertical plane containing sun rays, then shadows are parallel."*

- *"The shadows of the two sticks will be parallel only if the vertical plane of the oblique stick contains sun rays."*

The first two statements stand for two different ways of approaching the problem on the part of the students: the movement of the Sun and the movement of the sticks; the third statement makes explicit the uniqueness of the situation in which shadows are parallel.

After further discussion the collective construction of the two statements below was attained:

- *"If sun rays belong to the vertical plane of the oblique stick, shadows are parallel. Shadows are parallel only if sun rays belong to the vertical plane of the oblique stick "*

- *"If the oblique stick is on a vertical plane containing sun rays, shadows are parallel. Shadows are parallel only if the oblique stick is on a vertical plane containing sun rays"*

In order to help the students in the proving stage it was preferred not to express the statement in its standard, compact mathematical form *"if and only if..."* (its meaning in common Italian cannot be distinguished from the meaning of *"only if..."*) .

e) Preparing proof; the following activities were performed:

- individual search for analogies and differences between one's own initial conjecture and the three "cleaned" statements considered during the stage **d)**;

- individual task: *"What do you think about the possibility of testing our conjectures by experiment?"*

- discussion concerning students' answers to the preceding question. During the discussion, gradually students realize that an experimental testing is *"very difficult"*, because one should

check what happens *"in all the infinite positions of the sun and in all the infinite positions of the sticks".*
This long stage of activity (about 3 hours) was planned in order to enhance students' critical detachment from statements, motivate them to proving and make clear that since then classroom work would have concerned the validity of the statement "in general".

f) Proving that the condition is sufficient (activity in pairs, followed by the individual wording of the proof text);

g) Proving that the condition is necessary (short discussion guided by the teacher, followed by the individual wording of the proof text).

h) Final discussion, followed by an individual report about the whole activity (at home).

The following materials were collected: videotapes of the initial stages (**a** and **b**); tape-records of discussions and teacher-students interactions; all the students' individual written texts.The data which we are about to consider mainly concern stages **b)** and **f)**.

4. Students' behaviour

All students actively took part in the production of the initial conjecture. 29 students (over 36) were able to follow the following activities (from **c** to **h**) in a productive way.

For each type of students' behaviour one example of written texts individually produced by students during the stages **b)** and **f)** will be reported entirely. At this stage of the research we deem important to dwell on typical behaviour that can justify the plausibility of our hypothesis and to examine it more deeply (in view of its subsequent and more extensive confirmation).

It is possible to see how at the conjecture formulation stage there is much inaccuracy from the point of view of language, concerning in particular the expressions used to indicate a vertical plane containing sunrays. Through gestures with the hands or the movement of sticks it is clear that the students intend to indicate a vertical plane, but often they call it *"direction of rays"*. During the experiment this inaccuracy is gradually overcome: "concepts in act" (Vergnaud) receive appropriate names. Another aspect concerns the terms *"it can be seen"*, *"looking"*(referred to shadows): it is worthwhile to remember that no sunlight or spotlight was available in the class, therefore the students looked and saw with their imagination.

4.1. Correct conjecture with justification (21 students)

Underlining indicates traces of connections between conjecture production and proof construction.

Formulation of the conjecture with shifting of the stick:
(Beatrice) *"I tried to put one stick straight and the other in many positions (right, left, back, front) and with a ruler I tried to create the parallel rays. I sketched the shadows on a sheet of paper and I saw that: if the stick moves right or left shadows are not parallel; if the stick is moved forward and back shadows are parallel. Shifting the stick along the vertical plane, forward and back, the two sticks are always on the same direction, that is to say they meet the rays in the same way, therefore shadows are parallel. Whereas shifting the stick right and left the two sticks are not on the same direction anymore and therefore do not meet the sun rays in the same way and shadows in this case are not parallel. Shadows are parallel if the oblique stick is moved forward and back in the direction of sunrays."*
Proof: *"Shadows are parallel because, as we already said, sun rays belong to the vertical plane of the oblique stick.*
But all this does not explain to us why this is true. First of all, though the sticks stand one in an oblique and the other in a vertical position, they are aligned in the same way and if the

oblique stick is moved along its vertical plane and is left in the point in which it becomes vertical itself we see that they are parallel and, as a consequence, their shadows must naturally be also parallel, and also parallel with the shadow of the oblique stick, which has the same direction of that produced by the imaginary, vertical stick."

In this case the justification produced at the beginning*("meet the sun rays in the same way")* is the one that in the following proof makes Beatrice imagine the oblique stick moving along the vertical plane containing sun rays.

Formulation of the conjecture with the movement of the Sun

(Sara) *"They could be parallel if I imagine to be the sun that sees and I must place myself in the position so as to see two parallel sticks. In this way the sun sends its parallel rays to enlighten the sticks. But if the sun changes its position it will not see the parallel sticks and, therefore, their shadows will not be parallel either. Shadows can be parallel if the oblique stick is on the same vertical plane as the sun rays."*

Proof: *"If the sun sees the straight stick and the oblique stick parallel it is as if there were another vertical stick at the base of the oblique stick. If this stick is in front of the oblique stick its shadow covers the shadow of the oblique stick. These shadows are on the same line, therefore, the oblique and vertical sticks shadows are parallel."*

In this case the initial idea *"I imagine to be the sun"* seems to suggest the main argument of the proof (the shadow of the imaginary, vertical stick covers the shadow of the oblique stick).

Concerning production of the statement, Beatrice's and Sara's texts give evidence of complex mental processes correspondig to our hypothesis.

Concerning proof, both texts show interesting traces of the detachement from the problem situation (e.g.: *"I imagine to be..."* becomes *"If the sun sees"*) and the original statement. Students seem to be aware that it is necessary to validate the statement by a reasoning process (*"But all this does not explain to us why this is true."*). Many other texts show similar aspects.

We notice that in both cases above, just as for the majority of students, the dynamic process that brought to the production of the statement (movement of the sun or movement of the stick) is found again in the proving process. Yet the dynamic exploration implemented during the construction of the proof, though it shows remarkable similarities with the one implemented during the production of the conjecture as to the type of movement, differs deeply as to the function assumed in the thinking process: from a support to the selection and the specification of the conjecture, to a support for the implementation of a logical connection between the property assumed as true (*"vertical sticks produce parallel shadows"*) and the property to be validated.

4.2. Correct conjecture without justification (6 students)

6 students out of 36, be their level high or low, formulated the conjecture correctly, but during the formulation did not manage to produce arguments backing up their hypothesis. This fact seems somehow to affect the subsequent proof that turns out to be lacking in "arguments" and rather confused.

(Elisabetta) Conjecture: *"In some cases, although the oblique stick is in a position different from that of the vertical stick, the parallelism is kept, whereas in other cases the parallelism in shadows is not kept. Therefore, shadows can be parallel only if the oblique stick* [meaning with a gesture the vertical plane] *is parallel to the direction of the straight stick shadow, that is to the sun rays."*

Proof: *"Our statement is true because if the vertical plane of the oblique stick gathers the sun rays as that of the vertical stick, then the two shadows will be projected on the same line."*

4.3. Wrong conjecture (9 students)

9 students, be their level high or low, produce wrong conjectures probably suggested by the principle *"sun rays are parallel, then ..."* or by drawings that owing to their bidimensional nature may be misleading, and are also static and so they may stick the attention on particular situations.

(Vincenzo) Conjecture: *"In my opinion shadows cannot be parallel if the two sticks are one vertical and the other not vertical. I took the two sticks, I put them in a vertical position and shadows were parallel, then slowly I moved the right-hand side stick and noticed that its shadow moved. In my opinion they do not remain parallel, because if I have two vertical sticks, their shadows are parallel because rays are parallel, that is to say they come across the obstacle and form the shadow. But if I move slowly, rays that were hindered before now pass by, though they are hindered from another point, that is to say the shadow moves and, therefore, it is not parallel anymore."*

At the proving stage, after classroom discussions, 6 of these students "make up for" the lost grounds and it can be noticed how their proof is full of constructions and argumentations, as if these students had to reconstruct the conjecture to be proved:

(Vincenzo) Proof: *"The statement is true because: let us imagine to have an oblique stick and a vertical stick. Let us imagine to draw an imaginary line, perpendicular to the horizontal plane, starting from the point of the oblique stick. Let us do the same thing with the vertical stick but the other way round, meaning that I draw an imaginary oblique line parallel to the oblique stick.*

It happens that I get two vertical lines with parallel shadows and two oblique lines with parallel shadows. The imaginary stick casts a shadow into the direction of the oblique stick, as a consequence the shadows between the oblique stick and the vertical stick are parallel".

5. Conclusions

It appears to us that the data just illustrated are consistent and make our hypothesis plausible.

Actually, as concerns the production of the statement, argumentative reasoning fulfils a crucial function: it allows students to consciously explore different alternatives, to progressively specify the statement and to justify the plausibility of the produced conjecture (see 4.1.). On the other hand, students that produced wrong conjectures later show the need of reconstructing the valid conjecture in order to produce the proof (see 4.3).

The fact that poor argumentation during the production of the statement always corresponds to lack of arguments during the construction of the proof seem to confirm the close connection that exists between production of the conjecture and construction of the proof (see 4.2.).

Moreover, the consistency among personal arguments provided during the production of statements and the ways of reasoning developed during the proof seems to be confirmed:

- by the fact that the type of argumentative reasoning made during the production of the statement by one student is resumed by him/her (often also with similar linguistic expressions) in the justification of the statement subject to proof;
- by the fact that the kind of dynamic process (movement of the sun or the stick) recorded at the conjecture stage is almost always the same as the one used at the proof stage.

A further element surfaces during the teaching experiment: it can be observed that at the statement formulation stage the exploration by students almost always concerns both the parallelism and the non-parallelism, even if this process is not "abridged" (obviously, owing to the lack of experience in standard mathematical formulation) in a statement such as *"if and only if"*.

6. Discussion

As mentioned in the introduction, the hypothesis on which we worked seems to have important didactic implications, since it calls into question the traditional school approach to theorems.In fact, usually in Italy and in other Countries the teacher asks the students to understand and repeat proofs of statements supplied by him, which appears one of the most difficult and selective tasks for grade IX-X students. Only as possible last stage (often reserved to the top level students or students choosing an advanced mathematical curriculum) the teacher asks the students to prove statements, generally not produced by students but suggested by the teacher. Even more seldom students are asked to produce conjectures themselves. If our hypothesis is valid, during this traditional path students' difficulties can at least partly depend on the fact that they should reconstruct the cognitive complexity of a process in which mental acts of different nature functionally intermingle starting from tasks that by their nature bring them to partial activities that are difficult to reassemble in a single whole. Our teaching experiment suggests an alternative didactic path.

Just for the importance of such didactic implications we deem opportune to critically analyse some possible limits of the study made so far and to sketch further developments of it.

6.1. Critical analysis of findings and further research

First of all, we must consider in what sense students have performed a mathematical activity concerning theorems.

The object of the experiment is a hypothesis concerning the physical phenomenon of sunshadows; it has as a geometric counterpart, at the level of model, a statement of parallel projection geometry. Students produce their conjecture as a hypothesis concerning the phenomenon of sunshadows; when they verify their conjecture most of them seem to be aware of the fact that they must get the truth of the statement by reasoning, starting from true facts. Most of them produce a validation realized through a deductive reasoning. Actually their reasoning starts from properties considered as true *("two vertical sticks produce parallel shadows")* and gets the truth of the statement in the "scenary" determined by the hypothesis.

In this way, students produce neither a statement of geometry "strictu sensu", nor a formal proof: objects are not yet geometric entities, deduction is not yet formal derivation. But their deductive reasoning shares some crucial aspects with the construction of a mathematical proof. Moreover, the whole activity performed by students shares many aspects with mathematicians' work when they produce conjectures and proofs in some mathematics fields (e. g.: differential geometry): mental images of concrete models are frequently used during those activities. As to proof, mathematicians frequently come near to realize the ideal of the formal proof only during the final stage of proof writing. During the stage of proof construction, the search for "arguments" to be "set in chain" in a deductive way is frequently performed through heuristics, the reference to analogical models and keeping into account the semantics of considered propositions (cf Alibert & Thomas, 1991).

For these reasons we think that the activity performed during our teaching experiment may represent an approach to mathematics theorems which is correct and meaningful from the cultural point of view.

In our opinion, the continuity aspects hihglighted by us represent a huge potentiality for the development of the students' ability to prove conjectures; nevertheless, this potentiality needs an adequate educational context in order to surface successfully. In planning our teaching experiment we singled out some conditions that are probably necessary to this end; they concern:

- the didactic contract set up in the classroom (the production of a conjecture to solve an open problem, the value of an hypothesis as an "argumented choice");

- the didactic path in which the task is inserted (particularly, in our case, the choice of the field of experience of sunshadows as a long term learning environment);
- the management of classroom work after the task (individual activities alternating with activities in pairs and discussions; activities to prepare the proof stage - see e).

We are not as yet able to establish whether all the conditions that we singled out are actually necessary and sufficient for the extensive implementation of the process that we recorded in our teaching experiment.

It is necessary to ascertain what the actual weight of the didactic contract is, through comparisons with classes having a different history behind.

It is necessary to find out how much, and how, the cognitive unity of theorems appears also in mathematical fields other than geometry (and, in particular, that of "shadows geometry").

It appears also important to ascertain the consequences of "theorems cognitive unity" experiences on the activity of standard theorems proving, proposed through their statements.

Finally, it seems opportune to investigate the connections, the analogies and the differences between the procedures for the dynamic exploration of the problem solving situation during the production of the conjecture and, during the process of proof construction, the procedures for the dynamic exploration of the situation determined by the hypothesis.

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SOME DYNAMIC MENTAL PROCESSES UNDERLYING PRODUCING AND PROVING CONJECTURES

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The purpose of this report is the introduction and justification, on the basis of a teaching experiment, of a hypothesis concerning the crucial role that can be played by the dynamic exploration of the problem situation in the production and proof of the conjecture required to solve the problem. We will show how students can generate the conditionality of the statement and the functional connection with the subsequent proof through the dynamic exploration of the problem situation.

1. Introduction

Two previous reports had tried to focalize (through historic and epistemological analyses) the main cultural aspects of theorems in geometry (Boero & Garuti, 1994) and arithmetic (Garuti & al., 1995) in order to plan and analyse teaching experiments aimed at:

- verifying the possibility of productively involving students in the approach to theorems;
- identifying the difficulties found by students during that approach, the necessary mediating interventions on the part of teacher, etc.

The teaching experiments carried out showed how the students consciously took over the conditionality and generality of the statements (under the proper guide and mediation of the teacher) and then tried to prove them. However, the activity of most of the students greatly depended on teacher's interventions and their acquisitions were mainly based on the constructive proposals of a small number of their schoolmates.

This research project went on analysing mental processes underlying the production and proof of conjectures in mathematics. We believed that such analysis could give us some hints on suitable problem situations and the best class-work management modality for an extensive involvement of students in the construction of conjectures and proofs.

In particular, we took into consideration the conditionality of the statements, to which the logical structure of the proving process is connected. We have tried to formulate some hypotheses concerning the production of conditional statements and related proving developments. In order to do this, reference has been made to preceding studies, which suggested: the importance of the exploratory activity during the production of conjectures (cf. Polya's *"variational strategies"*; see also Schoenfeld, 1985); the relevance of mental images (as *"a pictorial anticipation of an action not yet performed"*, Piaget & Inhelder, 1967 - see Harel, 1995) in the anticipatory processes in geometry; the possibility of deriving the hypothetical structure *"if...then..."* from the dynamic exploration of a problem situation (cf Caron, 1979).

We therefore came to the following hypothesis referred to a didactic situation where students are requested to solve an open problem through the formulation and proof of a conjecture.The hypothesis concerns the crucial role that can be taken on by the dynamic exploration of the problem situation both at the stage of conjecture production and during the proof. The hypothesis is organised as follows:

- as to the conjecture production,

A) *the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, that will be subsequently detached from it and then "crystallize" from a logic point of view* ("if......, then.... ");

- as to the proof construction,

B) *for a statement expressing a sufficient condition* ("if...then..."), *proof* can *be the product of the dynamic exploration of the particular situation identified by the hypothesis;*

C) *for a statement expressing a sufficient and necessary condition* ("...if and only if..."), *proving that the condition is necessary* can *be achieved by resuming the dynamic exploration of the problem situation beyond the conditions fixed by the hypothesis.*

At this point we had to work out, put into practice and analyse a teaching experiment which let us explore the plausibility of the hypothesis, supplied the relative supporting elements and paved the way for further in-depth studies. In order to do this, reference has been made to:

- our previous observations made about the behaviour of students struggling with the formulation of hypotheses and conjectures in both mathematical and non-mathematical fields (cf Boero & al., 1995). Those observations stressed the importance of the choice of the context ("field of experience") as a crucial factor in order to activate mental processes of dynamic exploration of the problem situation;
- studies performed by Balacheff (1988), De Villiers (1991), Duval (1991), Hanna & Jahnke (1993), Mesquita (1989), Moore (1994), Tall (1995) and concerning approach to proof and epistemological, cognitive, pragmatic differences between argumentative reasoning and deductive reasoning. Our planning of the teaching experiment and the subsequent analysis were influenced by those studies: in particular, see 2.3. (stage e of the teaching experiment), 3.2 and 4.

The teaching experiment is described in § 2. The analysis and conclusions of the teaching experiment are shown in § 3. The discussion (§4) contains some reflections on our findings and indicates some of the developments suggested by our research.

2. Teaching Experiment

We tried to identify a suitable learning environment and proper tasks for the development of a production process of meaningful conjectures in classes with a suitable background. In addition, we tried to construct a teaching experiment which favoured (through the teaching activity succession and the observation system) the emergence and recording of the processes which our hypothesis refers to.

2.1. Learning environment

We have chosen the field of experience of "Sunshadows" as learning environment for our teaching experiment. The field of experience of sunshadows is a context in which students can naturally explore problem situations in different dynamical ways. In order to study the relationships between sun, shadow and the object which produces the shadow, one can imagine (and, if necessary, perform a concrete simulation of) the movement of the sun, of the observer and of the objects which produce the shadows.

The field of experience of sunshadows was chosen because it offers the possibility of producing, in open problem solving situations, conjectures which are meaningful from a space geometry point of view, not easy to be proved and without the possibility of substituting proof with the realization of drawings.

2.2. Classes and students' background

The teaching experiment was carried out in two grade VIII classes of 20 and 16 students, at the beginning of the third school year with the same teacher. Students had already interiorized the habit of producing argumented hypotheses in different domains (mathematical and non-mathematical), writing down their reasoning. Students had already experienced situations of statements production in arithmetic and geometry; they had approached proof production in the arithmetic field (see Boero & Garuti, 1994 and Garuti & al, 1995).

Concerning "Sunshadows", students had already performed about 80 hours of classroom work in this field of experience. They had observed and carefully recorded the sunshadows phenomenon over the year (in different days) and over the morning of some days. They had approached geometrical modeling of sunshadows and solved problems concerning the height of inaccessible objects through their sunshadows. In particular, students had already realized some activities which needed the imagination of different position of the sun and of the observer in order to produce hypotheses concerning the shape and the length of the shadows.

2.3. Classroom activities

In the two classes the activities were organised according to the following stages (whole amount of time for classroom work: about 10 hours):

a) Setting the problem :

"In the past years we observed that the shadows of two vertical sticks on the horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and an oblique stick? Can shadows be parallel? At times? When? Always? Never? Formulate your conjecture as a general statement."

(Individual work or work in pairs, as chosen by the students)

Some thin, long sticks and three polystyrene platforms were handed, in order to support the dynamic exploration process of the problem situation.

b) Producing conjectures: many students started to work with the thin sticks or with pencils. They started to move the sticks or to move themselves to see what happened. Other students closed their eyes. The absence of sunlight or spotlight in the classroom hindered the experimental verification of conjectures they were formulating: it was the mind's eyes that were "looking". Students individually wrote down their conjectures.

c) Discussing conjectures: the conjectures were discussed, with the help of the teacher, until statements of correct conjectures were collectively obtained which reflected the different approaches to the problem by the students.

d) Arranging statements: through different discussions, under the guidance of the teacher, the following statements,"cleaned" from metaphors and more precise from a linguistic point of view than those produced by students at the beginning, were collectively attained:

- *"If the sun rays belong to the vertical plane of the oblique stick, shadows are parallel."*
- *"If the oblique stick moves along a vertical plane containing sun rays, then shadows are parallel."*
- *"The shadows of the two sticks will be parallel only if the vertical plane of the oblique stick contains sun rays."*

The first two statements stand for two different ways of approaching the problem on the part of the students: the movement of the Sun and the movement of the sticks; the third statement makes explicit the uniqueness of the situation in which shadows are parallel.

After further discussion the collective construction of the two statements below was attained:

- *"If sun rays belong to the vertical plane of the oblique stick, shadows are parallel. Shadows are parallel only if sun rays belong to the vertical plane of the oblique stick "*
- *"If the oblique stick is on a vertical plane containing sun rays, shadows are parallel. Shadows are parallel only if the oblique stick is on a vertical plane containing sun rays"*

In order to help the students in the proving stage it was preferred not to express the statement in its standard, compact mathematical form *"if and only if..."* (its meaning in common Italian cannot be distinguished from the meaning of *"only if..."*) .

e) Preparing proof; the following activities were performed:

- individual search for analogies and differences between one's own initial conjecture and the three "cleaned" statements considered during the stage **d)**;
- individual task: *"What do you think about the possibility of testing our conjectures by experiment?"*
- discussion concerning students' answers to the preceding question. During the discussion, gradually students realize that an experimental testing is *"very difficult"*, because one should check what happens *"in all the infinite positions of sun and in all the infinite positions of the sticks"*.

This long stage of activity (about 3 hours) was planned in order to enhance students' critical detachment from statements, motivate them to proving and make clear that since then classroom work would have concerned the validity of the statement "in general"

f) Proving that the condition is sufficient (activity in pairs, followed by the individual wording of the proof text);

g) Proving that the condition is necessary (short discussion guided by the teacher, followed by the individual wording of the proof text).

h) Final discussion, followed by an individual report about the whole activity (at home).

2.4. Collected materials

The following materials were collected: videotapes of the initial stages (a and **b**); tape-records of discussions and teacher-students interactions; all the students' individual, written texts.The data which we are about to consider mainly concern stages **b)**, **f)** and **g)**.

3. Some findings

The teaching experiment analysis seems to confirm the validity of our hypothesis, as proved by the behaviour of the great majority of the students of the two classes. All students actively took part in the production of the initial conjecture. 29 students (over 36) were able to follow the activities (from **c** to **h**) in a productive way.
The elements found which confirm our hypothesis can be summarized up as follows:

3.1. As regards **A)** (relevance of the dynamic exploration on the problem situation during the conjecture production stage), the analysis of the videotape shows that at least one half of students (in the reality, probably more) performs the dynamic exploration of the problem situation in different ways: indicating with their hands the imagined movement of the sun, or moving themselves, or moving the oblique stick, or moving the platform supporting the sticks, etc.

On the other hand, in 14 individual texts (out of 36) there is explicit evidence of the passage from the imagined (and/or simulated) dynamic exploration of the problem situation to focusing on a temporal section, with successive transition to the formulation of a statement "crystallized" from a logic point of view:
(Simone)*"If we took into consideration two sticks, of which one vertical, the shadows shall be parallel when the two sticks are viewed parallel by the sun. If we suppose that the person is looking in the position of the Sun, by going round the sticks we can observe that the sticks are parallel in a certain position and the shadows are also parallel since the difference in position of the two sticks cannot be seen from that position. Thinking about the shadow space we can say that the non-vertical stick seems to be within the shadow space. Let's imagine an imaginary vertical stick representing the oblique one, in line with the sun rays and the same stick, the oblique one cannot be seen so it seems to be vertical, forming parallel shadows.*

The shadows can be parallel if the sun is situated along the direction of the oblique stick [with a gesture he indicates the vertical plane of the oblique stick]"

During the subsequent discussion, Simone explains how he produced this conjecture. Simone's gestures show that he moves the polystyrene plane supporting the sticks "at random" (notice should also be paid to the generality of his reasoning) after identifying himself with the sun. Then, he places a new stick (which he calls *"imaginary stick"*) in the same position he described in the written text, making the polystyrene plane rotate until the non-vertical stick is completely hidden by the "imaginary" vertical one. At this point he says *"well, now in this position the shadows are parallel because..."*.

Finally, it is interesting to analyse the way in which certain initially wrong conjectures are overcome: at the beginning of stage **b)** some students hypothesize that shadows are always parallel, on the basis of a kind of "principle" (according to their previous school experience): *"the sun rays are parallel, so they give parallel shadows"*. This conjecture is overcome by imagining and/or simulating the movements of the sticks or the sun. Those movements allow students to explore new alternatives. Here follows an example:

(Lucia) *"I think shadows are parallel because the oblique stick functions like a normal object perpendicular to the ground, so if the rays are equal for all the objects, the shadows will be parallel.*

(I've changed my mind)

By making a small model [they had fixed sticks to the desks with adhesive tape] *we found out that the parallelism of shadows depends on the position of the sun, that is, if we put the sun behind (or in front of) the sticks, the shadows are parallel but if the sun is placed on the side of the sticks then the shadows form an angle, spread apart and are no longer parallel"*.

3.2. As regards **B)** (relevance of the dynamic exploration of the situation determined by the hypothesis during the construction of the proof that the condition is sufficient), the following texts well represent the individual texts produced by most students:

(Giovanni) *"The sun "moves". At a given moment it "sees" the two parallel sticks and relative shadows. As the sun is far away it "sees" the two shadows parallel, so it imagines the oblique stick to be vertical (imaginary stick)* [introduced by Simone during the discussion phase]. *But if the imaginary stick were real its shadow would cover that of the oblique stick, that is they are on the same line. Well, now we know that the shadow of the two vertical sticks are parallel and at this moment it is as if we saw two parallel shadows because that of the oblique stick is "under" that of the imaginary one. Now, if we removed the imaginary stick, the shadow of the oblique stick would appear again since it was "under" the parallel shadow of the imaginary stick, so the shadow of the oblique stick is also parallel to that of the vertical one"*.

(Fabio) *"If we take two vertical sticks we know that their shadows are, of course, parallel. If we moved, that is inclined one of the two sticks along the vertical plane of the rays, the situation will not vary since the oblique stick along this plane seems to be another vertical stick, lower than the first. Consequently, their shadows are parallel"*.

(Daniele) *"Elisa and I have followed this line of reasoning: we have set two sticks straight knowing and seeing that the shadows were naturally parallel and we have tried to incline both sticks in the same direction* [they refer to the parallel vertical planes] *and found out that the shadows remained parallel even if the sticks were oblique. Then, we arrived to the conclusion that, since the sticks inclined along a vertical plane of sun rays are like two straight sticks but lower, so their shadows are parallel because they are like two vertical sticks. We concluded that the shadows of two sticks, one vertical and one oblique, are parallel if the sun rays are projected along the vertical plane of the oblique stick"*.

It seems to us that from these texts clearly comes out the fact that the dynamic exploration of the situation singled out by the hypothesis fulfils an important function in order to promote the logical connection between the property accepted as true (parallel sticks produce parallel shadows) and the property to be confirmed (shadows are parallel): the movement of the stick keeps the direction of its shadow (since it happens in the vertical plane containing sun rays) and, therefore, opens the possibility to reason in a transitive way (e.g.: the real, vertical stick produces a shadow parallel to the one of the imaginary, vertical stick; the oblique stick produces a shadow aligned with that of the imaginary, vertical stick; therefore the oblique stick produces a shadow parallel to that of the real, vertical stick). It also seems interesting to underline the fact that the hypothesis fixes the vertical plane on which the movement takes place that allows to relate logically the property to be proved with the property assumed as known. In this sense the dynamic exploration implemented during the construction of the proof, though it shows remarkable similarities with the one implemented during the production of the conjecture, differs deeply as to the function assumed during the thinking process: from a support to the selection and the specification of the conjecture, to a support for the implementation of a logical connection.

3.3. As regards **C)** (the dynamical exploration of the problem situation is resumed during the construction of the proof that the condition is necessary), we observe that:

- in some cases the sun or its rays are moved:

(Stefania): *"If the sun rays do not longer belong to the vertical plane of the oblique stick, the sun would "see" three sticks: one vertical, one oblique and an imaginary vertical one that casts shadow. Taking for granted that the shadows of the two vertical sticks are always parallel independently from the position of the sun or its rays, then, the sun would cast three shadows, of which two parallel and one oblique with respect to the other two. And if this shadow of the oblique stick were not aligned with that of the imaginary stick, it will be neither parallel with the shadow of the vertical stick, so the shadows would not be parallel and the hypothesis would not be true"*

- in other cases students moved the stick (beyond the vertical plane identified by the hypothesis):

(Sandra)*"In order to prove the second part of the statement* [the shadows are parallel only if the stick moves along a vertical plane containing sun rays]*we can move and place the oblique stick in another vertical plane so as to obtain two vertical planes, that of the oblique stick and that of the imaginary vertical stick. With this operation the two shadows are no longer situated in the same line so the shadow of the oblique stick and that of the vertical stick are no longer parallel. In this way, I've denied the previous statement so the shadows will be parallel only if the oblique stick is placed again along the vertical plane of the sun rays".*

4. Discussion

What relationship does it exist between our teaching experiment and producing and proving mathematical conjectures?

The object of the experiment is a hypothesis concerning the physical phenomenon of sunshadows; it has as a geometric counterpart, at the level of model, a statement of parallel projection geometry. Students produce their conjecture as a hypothesis concerning the phenomenon of sunshadows; when they verify their conjecture most of them seem to be aware of the fact that they must get the truth of the statement by reasoning, starting from true facts. Most of them produce a validation realized through a deductive reasoning. Actually their reasoning starts from properties considered as true *("two vertical sticks produce parallel shadows")* and gets the truth of the statement in the "scenary" determined by the hypothesis.

In this way, students produce neither a statement of geometry "strictu sensu", nor a formal proof: objects are not yet geometric entities, deduction is not yet formal derivation. But their deductive reasoning shares many aspects with the construction of a mathematical proof. Moreover, the whole activity performed by students shares many aspects with mathematicians' work when they produce conjectures and proofs in some mathematics fields (e. g.: differential geometry): mental images of concrete models are frequently used during those activities. As to proof, mathematicians frequently come near to realize the ideal of the formal proof only during the final stage of proof writing. During the stage of proof construction, the search for "arguments" to be "set in chain" in a deductive way is frequently performed through heuristics, the reference to analogical models and keeping into account the semantics of considered propositions (cf Alibert & Thomas, 1991).

In our teaching experiment, the "dynamic" learning environment of sun shadows was chosen in order to enhance the dynamic exploration of the problem situation on the part of students (keeping into account their background related to the same field of experience: see 2.2.). The great majority of the students (29 out of 36) has productively taken part in the statement construction and subsequent proof. This fact raises the problem of searching for learning environments similar or even more effective than that of the sun shadows as well as the problem of the transfert to "static" mathematics situations.

As regards the problem of finding suitable learning environments to develop the conjectures processes (dynamic exploration of problem situations), there are many learning environments which can be usefully compared with that of sun shadows (in particular, in the perspective of the "dynamic geometry" indicated by Goldenberg & Cuoco, 1995): Cabri or Geometric Supposer or Geometer's Sketchpad, even the "mathematical machines" and the "representation of the visible space" (Bartolini Bussi, 1995). Comparisons like these could propose different potentials and limits for the different learning environments.

With regard to the problem of the transfert from strongly contextualized theorems in a dynamic environment as that of the sun shadows geometry to the theorems of "context-free" mathematics, a number of confirmations derive from the observations that followed the teaching experiment in the two classes during activities with traditional geometry theorems. In particular, many students (of both high and average levels) could imagine the dynamic exploration of the geometric figures proposed for the formulation of conjectures and proofs. In the future, it will be necessary to make more systematic observations and establish comparisons with classes which have not carried out the activity described in this report (but have performed all previous activities).

A delicate matter concerns the variety of possible approaches to the conditionality of statements (and related connections with proving process). In fact, in Boero & Garuti (1994), a report dealing with the "Thales Theorem" and concerning the same learning environment of "Sunshadows", the following type of reasoning was identified in 3 students out of 34: *"The length of the shadows is proportional to the height of the sticks due to the parallelism of the sun shadows .... If the lines are parallel, the lengths of the segments cut on another two lines shall be proportional"*.The process appears to be very different from that considered in our hypothesis, since in this case the student passes from a recognition of causal dependency between parallelism and proportionality in the physical phenomenon, to the conditional statement that takes into account the possibility that lines cannot be parallel. This process asks therefore detaching from the physical phenomenon (that on the contrary can be deferred in the case of the approach to conditionality studied in this report). It is for this reason that we have formulated our hypothesis **A)** by emphasizing the possibility (*"can"*) that the conditionality of statements were originated in the dynamic exploration of the problem situation without excluding other possibilities. Further research will certainly supply interesting indications in this field, especially with respect to conjecture production processes more accessible to students in the approach to theorems.

Finally, we deem it important to examine deeply the implications of what surfaced during the teaching experiment as to the continuity that seems to exist between the way in which the dynamic exploration of the problem situation is attained and expressed and the way in which, during the proof construction, the dynamic exploration of the situation singled out by the hypothesis is in its turn attained and expressed. This continuity prompts the reflection on the holistic character that in an opportune educational context can be taken on by the process of theorems (statement and proof) production, apparently contrasting with the deep difference that exists between the argumentative reasoning needed to construct and make plausible the conjecture and the deductive reasoning to validate it (see Duval, 1991).

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TEACHERS' BEHAVIOURS IN TEACHING WITH COMPUTERS

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ABSTRACT

In this work we investigate, by means of interviews carried out with a sample of upper secondary school mathematics teachers, the behaviours of teachers when using educational software tools in their classrooms. In particular, we examine the different choices teachers have autonomously developed at this regard. We briefly outline the methodology of our work and we identify a number of issues we consider significant to investigate how teachers' behaviours in the use of computers influenced and was influenced by teachers' general behaviour and beliefs in teaching mathematics. Some findings resulting from the analysis of the interviews are presented according to the identified issues.

INTRODUCTION

In the recent literature in mathematics education a number of works is centred on teachers; different aspects are considered: education and training, behaviours in school practice, beliefs and conceptions, attitudes towards changes, etc..

Some contributions enlighten the fact that the teachers' role, although sometimes underestimated, is crucial in the implementation of curriculum reforms since they act as a filter between the curriculum developers and the students (Moreira & Noss, 1995; Törner, 1995). This is true both as regard changes in contents and in methods. A significant example in this context is that of the use of the computer in education, in particular the use of software tools in mathematics teaching. With the metaphore software tools we mean here not only educational packages but also other kind of software, such as, for example, spreadsheets, used for educational purposes. See (Love, 1995) for a discussion on software tools for mathematics education.

Initially computer based educational packages were developed mainly with the aim of an individualized course delivery for the student and they were often based on behaviourism (see, for example, the first CAI (Computer Assisted Learning) packages but also, in more recent times, some examples of ITS (Intelligent Tutoring Systems)). Two main aspects characterized this approach: the marginal role assigned to the teacher and the lack of initiative and autonomy left to the student in the interaction with the computer.

The substantial failure of the results obtained with these kind of systems, the consideration of different cognitive theories and their evolution, the technological and theoretical achievements in computer science had brought to the development of different kinds of computer based educational environments and to a radical re-evaluation of the role of the teacher. A different scenario in educational computing had prevailed. In this scenario both the student and the teacher play an active role. Educational software environments are considered as tools in the educational setting, tools which support both the student's learning process and the teaching process and tools which foster the interaction in classroom.

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In this paper we examine the behaviours of mathematics teachers when using educational software tools in their classrooms. In particular, here, we are interested in the choices and behaviours teachers have autonomously developed at this regard.

Studies exist considering the teachers' reactions to the use of educational software for mathematics. These studies are mainly focused on analysing the use of specific software by mathematics teachers and refer to experiences which are usually inspired and planned, also as regard teachers' training, by the researchers who perform the analysis. These studies are very important to acquire a deep knowledge on the possibilities offered by the software tools considered, but, in order to more widely explore teachers' behaviours, we think that it is interesting also to assume a different approach. We have considered situations in which software tools are used autonomously by the teachers in their classroom work independently from guided and controlled experimentations.

To perform our analysis we have considered teachers of upper secondary school (students' age from 14 to 18) who are carrying out the new mathematics curricula proposed by the Italian Ministry of Education. These curricula contemplate activities in computer laboratories and the use of software tools, but there are not precise suggestions on how to carry out these activities and on which tools to use. Teachers expressed, in general, personal preferences: for this reason we consider this context suitable for our investigation.

We note that this work originates from our previous studies on the way in which mathematics teaching has been affected by the introduction of informatics and on the conceptions mathematics teachers have developed about informatics and its teaching. See respectively (Bottino & Furinghetti, 1990; 1995).

RESEARCH METHOD

Sample

The sample consisted of 9 upper secondary school teachers using software tools while developing their mathematics programmes. The sample was chosen on the basis of the following parameters:
i) teachers working in the different categories in which upper secondary school is organised in Italy: lycei (scientific, humanistic, artistic), technical institutes (engineering, commercial, ..) and vocational schools; ii) teachers who have a certain degree of experience in the use of software tools; iii) teachers who use different kind of software tools. These parameters were identified in order to have a rather wide variety of teaching contexts, experiences and tools.

Procedure

The analysis on the teachers' behaviour in teaching with computers was performed by means of oral interviews. The interviews took place during the spring of 1995 in the afternoons when teachers were free from lessons. They were carried out without time limitation so that teachers could answer the questions without undue pressure. In each case the interviews lasted about three hours. The interviews were conducted by both authors together: one transcribed teachers' answers verbatim while the other took notes concerning impressions, observations, etc. The interviews were carried out on the basis of a written outline which was the same for all the teachers. This outline is intended as a preparatory and structuring tool to obtain a quite homogeneous set of data. In the preparation of this outline we refer not only to our but also to other previously developed researches (see for example Hoyles et Al, 1991). During the interviews, the different aspects related to the use of software tools in mathematics teaching were afforded in-depth considering the following elements: teacher' background; mathematics contents; software tools used; time dedicated to the use of software and its allocation throughout the year; organization of classroom and computer laboratory activities; contents afforded by means of software and aims pursued; methodology; teacher's role; teacher's estimation of the use of software; students' assessment performed (both qualitative and quantitative); teacher's attitude in mathematics teaching.

Analysis

In the analysis of the interviews our aim was to investigate how teachers' behaviours in the use of computers influenced and was influenced by teachers' general behaviour and beliefs in mathematics teaching. We based our analysis on the transcribed texts of the interviews and on the notes taken during their development. First of all we identified a number of issues we considered significant to perform the analysis according with the considered goal. The analysis was then carried out qualitatively considering these issues. Different approaches were used: individuation of information pertinent to each issue, cross-check, identification of analogies and pattern of behaviour; deduction of mutual influencing factors and so on. We note that the investigation on teachers' behaviours cannot be carried out by considering single factors in a cause and effect relationship; it is necessary to adopt a systemic approach in the analysis of the various factors and of their relationships.

The issues we identify as significant in performing the analysis are reported hereunder:

- 1) Why the teacher has decided to use software tools (motivations in the use of software).
- 2) Intentions that the teacher has in the use of software (both disciplinar and pedagogical).
- 3) Methodology.
- 4) Kind of interaction which has been established among the teacher, the student, the other students both in computer laboratory and in classroom.
- 5) Teacher's attitude and beliefs in mathematics teaching.
- 6) Influence of the context.
- 7) Evaluation given by the teacher about the use of software.
- 8) The relationship (if any) the teacher has established with colleagues, the research in mathematics education, the specialized press, etc.

In the following we present some findings from the interviews which give a first level analysis of the classroom scenarios established by the use of software.

SOME RESULTS

Results are presented according with the previously identified issues. Moreover, in table 1, we have summarized some information about the interviewed teachers which can be useful to better understand them.

1) **Motivations in the use of software**

   Different are the motives expressed (directly or indirectly) by the teachers for the use of software tools. There are superficial indications, such as: "because it is requested by the new mathematics programmes"; "because it is supplied by the maths textbook I used"; "because I followed a training course in which we were taught to use some software tools"; "because computers are attractive for students". There are indications related to teaching practice: "software is useful to speed the way in which some topics are afforded"; "software is useful to develop interesting exercises"; "drill & practice software are useful for students with relevant learning problems"; "software has a professional valour". There are indications which seem to be more strictly connected with teacher's views of mathematics teaching and its problems: "it is helpful to support conjecture activities"; "the use of software meets my renewal need"; "software can be considered as a sort of *super blackboard:* for example, it is possible to previously prepare graphs, tables, etc. so that students can be shown good level materials and it is not necessary to improvise at the blackboard"; "software is useful to enrich students' experience with mathematics concepts, especially students' capacity of visualization".

2) **Intentions in the use of software**

   The visualization of concepts, the introduction of topics, the possibility to carry out reinforcement exercise are the objectives of the majority of teachers. Some teachers indicate also the evaluation of students' learning, the application of maths concepts and a first approach to computers as objectives in the use of software; moreover the opportunity software gives of developing exercises which it is not possible to develop with paper and pencil is also indicated.

3) **Methodology**

   Different elements contribute to outline the methodology followed by the teachers in the use of software. All the teachers but one use software systematically (usually an hour per week). Software is used in laboratories usually equipped with 10-14 personal computers. Two teachers sometimes use the computer also in classroom with a data display. The majority of teachers follows a pre-planned didactic itinerary. Few teachers use reference materials (such as books or manuals); usually they prepare worksheets to be used by the students during computer laboratory activities. The work in the computer laboratory is organized in groups of two or three students. In general, the autonomy left to students is greater in laboratory than in classroom. We feel that sometimes this autonomy is only apparent: exercises to be developed with the computer are previously prepared in classroom; the teacher strictly co-ordinates groups work; etc. An interesting thing noted by one of the teacher is that his students are encouraged to divide among them the assigned task (e.g. for finding an area some students use the computer; some other use paper and pencil as a control; etc.).

4) **Social interactions**

In general teachers observe that during computer laboratory activities their interaction with the students is increased. Some of them perceive they role as changed: "I have to put myself under discussion"; "I think that for my students I can make errors using software while at the blackboard I have to be infallible"; "The teacher is no more the only reference point for the students, knowledge is shared, even if results obtained from the computer have to be considered in a critical way"; "For me the ideal would be to develop all my teaching in the computer laboratory". Few teachers assign importance to the interaction among students. One observe that, according to his experience, working with the computer does not foster students' interaction: the computer is seen as a substitute of the teacher. Moreover in some cases all the information seem to converge to the teacher. The use of software is seen by some teachers as less discriminative than other mathematics activities: "all students can do something"; "the one to one relation with the computer can be helpful for weak students and contemporary it offers to the teacher more ways to intervene". Some teachers compare their reactions to the computer with that of the students and judge this last one more positively.

5) **Teacher's attitude and beliefs in mathematics teaching**

Some teachers indicate as their general objective in mathematics teaching the fostering of students' mathematical thinking and the creative solution of problems. Some others have the objective that mathematics can be seen by students to be useful and applicable to the real world. The functions ascribed to mathematics vary from the buiding of real world models to the solution of problems and to the definition of an unambiguous language. Different are the motivations teachers convey to explain why students usually find mathematics difficult. There are motivations linked with their ideas about mathematics itself: "Mathematics requires fantasy and rigour"; "Mathematics needs a continuos control and the establishment of links between concepts". There are motivations linked with the way in which mathematics is taught: "Mathematics is often taught as a set of rules which have no meanings for the students"; "It is far from students experience".

An interesting element for the issue under discussion are the characteristics teachers ascribe to 'good' and 'weak' students. Some teachers indicate general characteristics such as intuition, autonomous re-elaboration and flexibility for 'good' students and rigidity and repetitivity for 'weak' students. Other teachers indicate characteristics which seem more linked with their actual teaching. For 'good' students: the capacity to exploit all the resources available; the capacity of visualization; the capacity of communicating with the others; the capacity of producing conjectures and the capacity of using computers. For 'weak' students: the difficulty of using the mathematics language and the difficulty in solving problems.

6) **Influence of the context**

The context (type of school, training followed, etc.) seems to influence teachers' behaviours as regard both contents and software used, especially the behaviuor of teachers of technical and vocational Institutes. In some cases, it seems to influence also teachers' expectations to students. Moreover the context seems to influence teaching orientations and aims: for example, humanistic lycei teachers show more attention towards conjecture and proof activities.

7) **Evaluation given by the teacher about the use of software**

All the teachers give a positive evaluation of the use of software tools. Some teachers state to have adapted their teaching objectives to the software used. A teacher observe that the use of software is amusing for the students and also for herself. General observations made are as follows: "software requires a great deal of work to the teacher to learn it"; "it can be useful to attract students"; "its use in classroom needs a lot of time"; "what is required by the computer is usually better accepted by the students than that asked by the teacher"; "software favours the communication between the students and the teacher"; "software can create experience in the use of mathematics concepts"; "it can induce a sort of laziness in some students". A teacher observes that there are reasoning methods which are proper of a person (e.g. abstract reasoning or intuitive approach by means of visualization) and the more or less success of a work methodology, such as, for example, the use of software, depends on these personal inclinations.

8) **Relationships established by the teacher**

The majority of the teachers have few relations with their colleagues. Only three teachers state to work in collaboration with another teacher and to periodically confront the results. Two teachers prepare the software exercises together with the laboratory technician (which is present only in some situation). Some teachers belong to groups which work in the field of mathematics education. Few teachers read specialized press.

CONCLUSION

The investigation performed outlines a scenario which can give a useful contribution to the investigation on teachers' conceptions about mathematics teaching and related changes. Questions for which it is possible to infer some answer could be, for example: Is software perceived as a real methodological change in mathematics teaching? Does its use contribute to create in the teacher a different image of the educational setting? Does the integration of the use of software in the mathematics curriculum contribute to the establishment of a different social interaction inside the classroom? Is the use of software guided by a previous and well established conception of mathematics teaching which is not changed by this use or the use of software put into discussion previous conceptions? How much is heard the necessity of change and how much it is supported by the use of software? Can the use of software in some cases hide a reluctance to change? and so on.

The development of our research will try to answer to these questions in order to study the conditions and the methodologies which affect teaching changes.

Teachers	Type of school	Software tools used(1)	Maths subjects in which software is used	Examples of exercises
1	Commercial Technical Institute	Derive Lotus Pascal	Algebra Calculus Statistic Financial maths	Approximation of curves (Taylor); computation of areas; simplex method
2	Engineering Technical Institute	Lotus MathCad Derive Graphic Calculus	Geometry Algebra Calculus	Geometrical transformation; matrices; linear systems; visualization of the convergence of a series
3	Humanistic Lyceum	Lotus Derive Cabri	Geometry Algebra	Definition of geometrical properties; formulas to generate prime numbers; graphs of functions
4	Professional Institute	Lotus Educational programs for drill & practice	Algebra	Graphs of functions; solution of equations; simple linear systems
5	Scientific Lyceum	Microcalc Lam	Geometry Algebra Calculus (few)	Geometrical transformation; truth tables; equivalence relations
6	Artistic Lyceum	Lotus QBasic Derive Gra-Fun	Algebra Analytical Geometry Statistics Probability Calculus	Algebraic transformations; study of functions; programming of algebraic formulas
7	Experimental Humanistic Lyceum	Lotus Turbo Pascal	Geometry Algebra	Geometrical transformation; variables; parameters; numbers properties
8	Scientific Lyceum	Microcalc Lam	Geometry Algebra Calculus	Disequations; transformations; truth tables
9	Humanistic Lyceum	Cabri Derive Microcalc Pascal Pocket calculators	Calculus Geometry Probability Algebra	Geometric conjectures and definitions; computation of areas; algebraic manipulation; simulation of probability frequencies; visual definition of trigonometric functions

(1) Lotus is a spreadsheet; Pascal, Turbo Pascal, QBasic are general purpose programming languages; Derive and MathCad are symbolic manipulation system; MicroCalc is a system for graphing functions in two or three dimensions; Cabri is a system for plane geometry; Lam is a system for geometrical transformations and for graphing functions; GraFun is a system for graphing functions; Graphic Calculus is a system for calculus. For further information see References.

Table 1: Some information about the interviewed teachers

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A DECIMAL NUMBER IS A PAIR OF WHOLE NUMBERS

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The purpose of this paper is to report some results of a larger project designed to develop a collection of diagnostic test instruments. The tests, together with an in-service training package are meant to be used to as a starting point for the teaching of key concepts throughout the curriculum. The paper explore the beliefs held by pupils of 11, 13, and 15 years related to the misconception that a decimal number consists of a pair of whole numbers separated by a decimal point. Mention will also be made of how some pupils conceive the decimal point as having the same role as the fraction bar.

Introduction

We regard the assumption that a decimal number is composed of two whole numbers to be the most important underlying misconception linked to the conceptual knowledge of decimal numbers. In Norway a decimal *comma* is used in the notation. Referring to Sackur-Grisvard & Leonard (1985), Resnick et al. (1989) analysed how children in different countries applied different rules when comparing decimal numbers. Resnick et al. argue that there are two main sources of the errors observed, named *whole number* errors and *fraction* errors. In addition to addressing Resnick's whole number error with regard to a wider set of items, this paper will describe how the children perceive the decimal point as a separator between two whole numbers. The discussion is based mainly on data from the KIM project, initiated and funded by the Norwegian Ministry of Education. The objectives of the project are to:

- develop a collection of test-instruments of a diagnostic nature which can be used as a starting point for teaching practice within various areas of school mathematics
- develop an integrated test and in-service training package that can be used by teachers as part of their assessment practice
- survey attitudes and conceptions that students have with regard to mathematics and the teaching of mathematics
- report the whole spectrum of student performance within the various areas of school mathematics, not only minimal competence
- survey student performance in relation to a broad spectrum of objectives specified in the current curriculum.

The project concerns mathematics from grades 1-9 7 to 15 years), and is intended to be extended also to grade 12, and is expected to run for several years.

Background

One of the difficulties of operating with decimal numbers is that they have several different meanings. The natural numbers are used mainly for counting objects or measuring units, while decimal numbers can be interpreted concretely in many ways, all of which occur in everyday life applications, for example as a result of a division, a part of a unit, a comparison of size or a point on the number line.

For each extension of the number system many children encounter a critical phase in their learning of mathematics. They have to become competent with a new symbol system in the context of the ideas of the previous systems. In addition, this extended knowledge of the symbol system is accompanied by an extension of the meaning of the numerical operations. It is therefore a significant advance in pupils' knowledge of mathematics to become competent in the concept of decimal numbers. Hiebert and Wearne (1986) claim that many pupils have trouble selecting features of whole numbers that can be generalised to decimal numbers, they often overgeneralise procedural features which do not apply to decimal numbers.

Pupils have experienced decimal numbers in connection with measurement of different kind, long before such numbers become part of instruction in school; they have some syntactic knowledge of decimal numbers which can become an obstacle to conceptual understanding. In almost all such applications of measurement, the decimal point can be regarded as a separator between different units of measure (m and cm, kg and g, pounds and pence etc.), and it *is* a *whole number* of pounds to the left of the decimal point and a *whole number* of pence to the right. It seems that the teaching of decimal numbers as *one number* which *can* contain *tenths, hundredths, thousandths* etc. of an unit, does not replace this first decimal experience with money and measurement. Often children learn decimal position values only as verbal labels for isolated digits. Teachers regularly claim that their pupils manage to solve arithmetic problems involving decimals correctly if money is introduced as a context to such problems. Thus they fail to see that the children do not understand *decimal numbers* in such cases, but rather that such understanding is not needed; it is possible to continue to work as if the numbers are whole, and change one hundred pence into one pound if necessary. It is doubtful whether a continued reference to money will be helpful, when it comes to developing understanding of decimal numbers; on the contrary, this can be a hindrance to the development of a robust decimal concept. Employing a comma in the decimal notation in Norway may induce pupils to look upon the decimal comma in a similar way as it is used to list items in a written text, which may strengthen the idea of the decimal comma (point) as a separator.

Several studies of the teaching of decimal numbers have been carried out. Bell, Swan & Taylor (1981) and Swan (1983) conducted teaching experiments which focused on intensive work with basic misconceptions linked to the decimal numbers. Wearne & Hiebert (1988) and Wearne (1990) tried in their teaching to get children to connect meaningful referents with decimals numbers. Streefland (1991) used the number line extensively in the teaching of fractions and decimal numbers, as did Mahrer, Martino & Davis (1994). Lachance & Confrey (1995) claims that many teaching experiments have treated the teaching of decimals as distinct and separate instructional units, and reports on results from an experiment involving instruction of decimal fractions embedded in a unique curriculum. The in-service part of the KIM project offers activities with the intention to focus intensively on the most common misconceptions.

Method

National data related to conceptual understanding of, and operations with, decimal numbers was collected by two written diagnostic tests, each taking one 45 minute teaching period. In total 104, 107 and 92 classes in grades four, six and eight (average age 10 years 7 months, 12 years 7 months and 14 years 7 months while tested) took part in this national standardisation. From these classes the papers from pupils with specific dates of birth in each month were analysed. This produced the sample size of 512, 510 and 519 pupils for the three year groups respectively. In total the children were asked to respond to 72, 95, and 99 questions respectively for the three grades. Several items were given to the all year groups.

Main findings and discussion

The following response pattern (Table 1) emerged from three items, asking children to choose the largest from three given numbers. In item 20 the pupils were asked to explain why they thought that the chosen number was the largest.

	Correct			Longest is largest			Shortest is largest		
13 Which of the following numbers has the largest value?		*3.75*			*3.521*			*3.6*	
	20	64	88	74	30	6	5	6	5
14 Which of the following numbers has the largest value?		*4.7*			*4.008*			*4.09*	
	31	72	94	35	9	2	32	18	4
20a Which of the following numbers has the largest value?		*0.87*			*0.649*			*0.7*	
	22	62	83	66	26	7	8	10	9

Table 1 Selected items comparing the value of given decimal numbers. Distribution in percentages per grade.

In their reasons for choosing the number with the longest decimal fraction (item 20b), children in most cases refer to the number to the left of the decimal point as a whole number. APU named this misconception "decimal point ignored"; from my point of view it is not ignoring the decimal point that is crucial but rather the idea of "pair of whole numbers" (pwn). Items 13 and 20a shows a similar pattern of responses, indicating that there is no differences in the children's strategies used in comparing decimal numbers to those used in comparing decimal fractions. For the three year groups, respectively 96, 86, and 65% of the pupils who chose 0.649 in item 20a also answered 3.521 to item 13, and similarly 93, 89 and 78% of these chose 0.5 as the smallest of the numbers in item 19a (see below), which confirms the stability of this idea. We notice an extensive amount of answers related to pwn for the younger children, and that this error gradually diminishes. Swan (1983) and Greer (1987) asked

11-12 and 12-13 year old pupils respectively to choose the largest of 0.62, 0.236 and 0.4, with the following response pattern, Swan: 17%, 50%, 28%, Greer: 45%, 12%, 43%. The proportion of "pwn-answers" is similar to our study. A similar response pattern as in the previous data was found when pupils were asked to ring the smallest of the numbers: 0.625; 0.25; 0.3753; 0.125 and 0.5.

19a Which of the following numbers has the smallest value?	Correct: *0.125*	Shortest is smallest: *0.5*	Longest is smallest: *0.3753*
	16 55 79	64 26 7	8 13 10
		0.25 (Other)	*0.625* (Other)
		8 13 10	1 1 1

Table 2 Item 19a comparing the value of given decimal numbers. Distribution in percentages per grade

This item was used in APU, Foxman (1985) for 15 year old pupils, with the distribution: 43%, 13%, 36%, 2%, 4%. There is a substantial difference between APU and the Norwegian responses with respect to the proportion of correct answers as for the pwn error. Item 14 function differently, since the whole number error will have different effect on the choice of answers. Comparing the whole numbers 7, 09 and 008 most children will know that 09 is the largest. Another item required the children to write down the next two numbers in the sequence 0.3; 0.6; 0.9. 52, 40 and 26% respectively of the children in each year group gave the answer 0.12; 0.15.

Resnick labelled sources for the errors like 3.6 (item 13), 0.7 (item 20) and 0.3753 (item 19) "fraction errors", APU used the expression "longest is smallest" (ls) for this misconception. Norwegian pupils explain their (ls) choices by writing "0.7 is larger than 0.87 because in 0.7 we have tenths and in 0.87 hundredths, and hundredths are smaller than tenths" or "when it is hundredths the numbers are more split up". Notice the much higher occurrence of fraction errors in the British responses. One reason for this could be the different emphasis on fractions in the two curricula. Contrary to pwn the percentage of the ls error is about the same for all year groups in the Norwegian sample, which indicates that this misconception is not focused on in the teaching of decimal numbers.

The two following items also require the pupils to compare the value of decimal numbers, but this time in different situations to the previous ones. To the item, given to grade 4:

21. Olav got the answer 4,9 and Lise 4,90, solving a math problem.

a. Is there any distinction between these answers. b. How did you arrive at your answer in a.?

21% answered that there is a distinction, because "90 is more than 9", or similar. The responses 7 and 0.43 or 43 (Table 3) to problem 18 below also indicate a conception of decimal numbers as a pair of whole numbers.

18. Write the missing number *5,47 = 5 + 0,4 +* ☐

To a larger extent than the previous items, this item reveals difficulties in understanding decimal notation, and also a higher proportion of pwn responses than in the previous problems which are more familiar to the pupils .

	0.07	7	0.43 or 43
Grade 4 (11y)	11	30	13
Grade 6 (13y)	39	22	10
Grade 8 (15)	66	9	8

Table 3 Response pattern to item 18. Distribution of correct and two common types of incorrect answers, in percentages per grade

The responses to the items below (Table 4) show that the pwn misconception is employed by children when adding decimal numbers. The numbers involved are chosen so that an algorithmic procedure should be unnecessary, and are presented in different ways for the same reason.

38 Add 0.1 and write the answer. a 4,256 b 3.9 c 6,98 d 5.4 e 7.03
32 Write the answer to 5.1 + 0.46 = ...
42 Write the number which is 0.01 greater than 53.724

		Grade 4 (11y)	Grade 6 (13y)	Grade 8 (15y)
38a	4.257	41	20	5
38b	3.10	24	12	3
38c	6.99	47	27	12
38e	7.04	40	24	9
32	5.47	39	18	6
42	53.725	38	15	10

Table 4 Percentage of pwn errors in percentages per grade for selected addition items

The analysis above has focused on some difficulties pupils have in understanding the symbolisation of decimal numbers. It is important to know that the number 0.437 has the value of 4 tenths plus 3 hundredths plus 7 thousandths, and that this is the same value as 437 thousands. In Table 5 the responses to two items related to the understanding of the decimal position system are given.

5. What does the digit 7 mean in 0.573? 70 7 0.7 0.07
6. Which digit is in the hundredths place in 6.423? 6 4 2 3

Item 5	Grade 4	Grade 6	Grade 8	Item 5	Grade 4	Grade 6	Grade 8
70	26	54	73	2	10	31	47
7	40	23	9	6	15	8	4
0.7	11	10	4	4	65	55	39
0.07	13	10	10	3	4	5	8

Table 5 Distribution in percentages per grade

It is reasonable to believe that this unexpectedly high proportion of incorrect answers can be explained by the way children normally read decimal numbers. 0.573 is in Norway usually read as "nought point five hundred and *seventy* three", and accounts for the answer 70 in item 5, those who answer 4 to item 6, do this because they read four *hundred* and twenty three.

When children first experience decimal numbers, usually in connection with money and measurement, they may be led to believe that the decimal point is introduced to separate two units of measurement. From the teaching of fractions they know that the fraction bar is used to split for example a part from a whole. They are also told that there is a relationship between fractions and decimal numbers. It is therefore not a great step further for them to conceive the decimal point as a separator also. A few children in grade 4 and 6 (6% and 4%) answered 5,10 or 10,5 to item 15a; indicating that the decimal comma is used as an separator between nominator and denominator of the given fraction. Table 6 shows the distribution of this error for the other questions in item 15 together with the most common incorrect responses in each case.

15 Write the fractions as decimal numbers: a) Five tenths b) Three hundredths c) Eleven thousandths d) Eleven tenths e) Two fifths f) One third.

	b) 0.03	0.3	Sep	c) 0.011	0.0011	Sep	d) 1.1	0.11	Sep	e) 0.4	0.2	Sep	f) 0.33	0.3	Sep
Grade 4	22	26	15	15	6	11	19	43	7						
Grade 6	55	16	6	37	18	4	35	43	4	12	18	11	15	20	14
Grade 8	75	12	3	47	30	2	41	50	1	27	14	13	38	14	12

Table 6 Correct, most common wrong response and reponses which indicates that the decimal point (comma) is regarded as a separator

It is interesting to notice that this error becomes more frequent as the familiarity of the equivalence between a given common fraction and its decimal representation decreases. Several children with a vague decimal number concept regress in such cases to more primitive ways of representing the concept.

We also observe the high proportions of the most common wrong responses to item 15 b, c and d and how permanent these are for all grades. It is especially interesting to notice that the answer 0.0011 becomes more widespread for older pupils. Interviews indicate that these pupils believe that since thousandths are involved, they first have to write two ceros to the left of the decimal point and then the value of the nominator. Even though older pupils perform better than younger it is interesting to notice that many of those giving an incorrect response tend to converge towards certain errors.

Three items 8, 16, and 31 require pupils to represent a shaded area (volume) of a figure by a decimal number. Reponses to these items show an extensive use of the decimal comma as a separator. In item 8, 16 out of 100 squares, arranged as a ten by ten square, are shaded. No alternatives answers are given. Item 16 show a five by four rectangle with eight out of the twenty squares shaded (a column of four squares and a two by two square). Four alternative choices for the response are listed: 8.12; 0.4; 8.20 and 0.8. In item 31, a picture of a graduated cylinder is shown. Two out of five parts of the cylinder is filled, indicated with a shading. This item have the choices 2.5; 0.4; 2.3; 0.2 for the correct response. Item 8 were used only in grade 4, and 12% gave one of the following answers 16,100; 16,84 or 84,16, representing the shaded part of the square by a decimal number, which is, in their mind, a pair of whole numbers made up by the shaded part, and either the whole square or the nonshaded part of the square. Table 7 shows the distribution of the responses to the two other items.

Item 16	Grade 6	Grade 8	Item 31	Grade 6	Grade 8
0.4	16	39	0.4	11	31
8.12	19	13	2.3	24	21
8.20	35	25	2.5	21	15
0.8	28	22	0.2	36	26

Table 7 Distribution of responses to items 16 and 31 in percentages per grade

In a an analogue way as explained for item 8, also older pupils use the decimal comma as a separator between the shaded and either the whole figure or the remaining nonshaded part of it, and believe that this is the appropriate decimal number.

The main purpose of the KIM project is to help teachers to put more emphasis on conceptual development, by highlighting common misconceptions and conceptual obstacles for key concepts in mathematics through diagnostic test and teaching material connected to the tests. Following a quote from Ausubel: *If I had to reduce all of educational psychology to just one principle, I would say this: The most single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly.* As educators we will have to help teachers decide *if* the pupils know, and *what* they know if they do not know what they ought to know. This paper have tried to shed light on one portion of what pupils ought to know about decimal numbers.

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THE STORY OF SILENCE:
TEACHER AS RESEARCHER - RESEARCHER AS TEACHER

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In Brown (1995a) a theoretical perspective of 'levels of strategies' was introduced to interpret how 'effective' mathematics teachers work. This case study illustrates the application of this theoretical perspective with one teacher in his second year of teaching to consider how the researcher and teacher work together (1) to identify purposes to be used as organising principles (filters, Brown, Hewitt, Mason 1994) to focus observations and actions - in this case 'using teacher silence in the mathematics classroom' and (2) to uncover implicit beliefs (Claxton, 1984, 1989) both underpinning and inhibiting (Brown 1995b) their development. The use is made of stories (Bateson, 1979, Bruner, 1990) in the writing of this paper to allow new insights to coemerge (Edwards and Núñez, 1995, Reid, 1995, emergent, Varela, Thompson, Rosch 1995) and to give a sense of the merging frames of reference of teacher and researcher.

Theoretical background

In Brown (1995a) an adaptation of work on a hierarchy of learning strategies (Nisbet and Schucksmith, 1986) was introduced to interpret how 'effective' mathematics teachers operate. This hierarchy was used as a tool in the training of prospective teachers:

> '... the central strategy (refers) to teachers' images of mathematics and mathematics teaching, giving an overall sense of direction to their work. Such philosophical and attitudinal perspectives (implicit learning theories and theories of self, Claxton, 1984, 1989) build up over time and are certainly not easily transferable, but do inform the decision-making necessary to apply lower order strategies. Next I associate macro-strategies with the teacher's purposes. For a particular purpose (eg gaining access to pupils' thinking) the teacher often has a range of behaviours which can be used at differing times and in differing circumstances. Micro-strategies are identified with these specific behaviours.' (Brown, 1995b)

The micro-strategies might be easily transferable as behaviours to trainee teachers who would still need to work at the level of purpose to begin to integrate the behaviours into a range of strategies. They will only recognise the micro-strategy as being useful if it conforms to their developing central strategy. For an illustration see Brown (1995a).

In the work of Rosch on basic-level effects in Lakoff (1985) basic-level categories are 'the generally most useful distinctions to make in the world (p49)' and have a middle position in that hierarchical model similar to purposes. The 'interactional properties' (p51), the lowest point in the hierarchy, which form 'clusters' are 'the result of our interactions as part of our physical and cultural environments given our bodies and our cognitive apparatus'. The obvious parallel here would seem to be with the particular behaviours, or micro-strategies, which an experienced teacher uses to achieve their purposes. These behaviours seem clustered in that there exist more than one micro-strategy available to be used for any given purpose. 'Superordinate categories', in the Rosch hierarchy, are not so embodied, however, and 'seem not to be characterised by images or motor actions' (p51). This fits with the way in which the central strategies of the

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teachers are tied to their implicit beliefs and are more difficult to work with since often they are unconscious. The intention became to develop ways of working with trainee teachers so that they could find their basic-level categorisations, or purposes in my model, which they might be able to use as filters (Brown, Hewitt, Mason 1994) to develop a range of effective (for them) micro-strategies with a consequent adaptation of central strategy where appropriate.

When working with teachers who wish to develop their practice there is a power in the use of story. This is both in the sense offered by Bateson (1979):

> ' A story is a little knot or complex of that species of connectedness which we call *relevance*... we face connectedness at more than one level: first, connection between A and B by virtue of their being components in the same story and then connectedness between people in that all think in terms of stories. Context and relevance must be characteristic not only of all so-called behaviour (those stories which are projected out into 'action'), but also of all those internal stories... I offer you the notion of *context* of *pattern through time*.' (p13-14)

and of Bruner (1990) who talks of matters which interest him particularly in the 'organisation of experience' and 'the role of narrativised folk psychology':

> 'One of them is usually called *framing* or schematising ... Framing provides a means of 'constructing' a world, of characterising its flow, ... what does not get structured narratively suffers loss in memory. (This is not) merely a matter of the laying down of traces and schemata within each individual brain ... Shotter insists very strongly that framing is *social*, designed for the sharing of memory within a culture.' (p56)

One strategy used to try to encourage patterning in the identification of macro-strategies or purposes with trainee teachers is to encourage them to tell anecdotes or 'brief-but- vivid' descriptions (Mason, 1994) of details of their practice in written or spoken form and to see what arises in terms of what is the same or different about them.

There follows a case study of work with a teacher in his second year of teaching to illustrate both the use of the model of hierarchy of strategies as a theoretical perspective and of the professional development technique of 'staying with the story'. The case study is written as a story, a sequence of 'vignettes, snapshots or perhaps a mini-movie, of a professional at work' (Miles, 1990) on which we reflect in a commentary after each vignette in the sequence. In this case, however, there is insight into two professionals at work as their frames merge over time.

The Case Study

1) Beginnings - separate frames - teacher (Alf) - researcher (Laurinda)

Alf: Reflecting back on the first year of teaching had produced a feeling of inadequacy akin to despair - looking back over all that time, looking for the lessons which had been 'good' from which to start to build next year they had seemed rare. No lesson really seemed to match up to my ideal image of what seemed possible and there was a strong sense of a gap between where my philosophy lay and the day to day practice of what was actually happening in the classroom.

Laurinda: Listening to these expressed thoughts, especially 'the gap' had offered the opportunity to work with dissonance (Brown, 1995b) a strong sense of feeling uncomfortable, using the theoretical perspective of central, macro- and micro-strategies. Working with experienced and 'effective' teachers over many years to get them articulating the detail of their practice what would it be like working with someone (not trained by me) so early in their career to see whether any of this was transferable as a model for professional development?

Alf and Laurinda: There was a discussion of the possibility of working together in the classroom. Alf said: *what do you want to do?* and the only answer was that if the work were to take place the agenda would emerge from conversations.

Commentary 1

Laurinda: What seemed crucial was that the agenda for the work was Alf's. My investigation would be subordinate to this agenda.

Alf: The question: *what do you want to do?* was asked from concern that there was not enough of interest in my classroom to warrant such time and attention. I was checking out that Laurinda didn't have unrealistic expectations of what might be going on.

Laurinda: This was unlikely because from my previous work I was aware that the way teachers talk about and describe their work within groups outside their own classrooms rarely gives insight into their current practice. It is as if they talk in terms of vectors with a direction of movement given by what they're working on which gives no sense of a relative position. I had few expectations of what Alf's classroom would be like.

2) The emergence of the 'it'

Travelling in a car, with Alf's attention partly taken up by driving, Laurinda asked whether he could bring to mind particular moments or times during a part or parts of lessons which had felt closest to his ideal.

This provoked two 'brief-but-vivid' (Mason, 1994) anecdotes:
Anecdote 1: During an A-Level lesson on partial fractions I was going through an example on the board, trying to prompt suggestions for what I should write. Some discussion ensued amongst the students, which ended in disagreement about what the next line should be. I said I would not write anything until there was a unanimous opinion. This started further talk and a resolution amongst themselves of the disagreement. I then continued with the rule of waiting for agreement before writing the next line on the board.
Anecdote 2: Doing significant figures with a year 9, I wrote up a list of numbers and got the class to round them to the nearest hundred or tenth, ... Keeping silent, I wrote, next to their answers, how many significant figures they had used in their rounding. Different explanations

for what I was doing were quickly formed and a discussion followed about what significant figures were.

Without any prompting from Laurinda there was suddenly an energetic statement of 'It's silence, isn't it? It's silence.'

Commentary 2 - frames and looking through them

Although we went on to identify more labels we had found our agenda - we could work on silence. The question remains how did we recognise 'silence' as something we could both work on? The frames begin to merge. Laurinda recognises the labelling of silence by Alf as the identification of a macro-strategy or purpose within her theoretical perspective. Alf was aware that his silence had forced students to think for themselves about what he was doing, putting the onus of explanation on them. He also became aware of 'silence' as a potentially broader categorisation. We were looking in the same direction with more of a sense of each other's frames. Micro-strategies involving silence and lesson introductions for algebra, given Alf's scheme of work, were discussed drawing on Laurinda's observations of effective mathematics teachers and lessons and Alf decided what he was going to teach.

3) Life histories of silence

We each wrote our own story of silence in the style of moments or incidents that came to mind and then shared the writings with a view to offering a sentence of the connections or related themes found in the other's. What follows are those sentences to give an indication of the difference in frames and yet broad experiences which were available at the start of the work together in the classroom:

Alf on Laurinda: Silence painful and angry as a child - alienation transformed into listening through groupwork and counselling training at University.

> 'Later Mum told me that Dad was just as upset as I was, but for different reasons. He got frustrated at not being able to give me any answers to my questions.' (Laurinda 20/11/95)

Laurinda on Alf: Travel - opening of the mind through contemplation of natural phenomena and other peoples and religions.

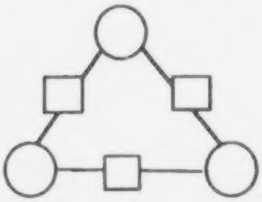
> 'Silence is also in waiting. I think I learned how to wait in Zimbabwe - hours by the side of roads waiting for buses and lifts. I have an image of waiting with Zimbabwean friends who would just sit ...' (Alf 20/11/95)

Commentary 3

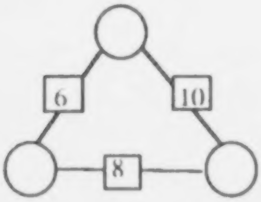
A common strand that emerged was an interest in and experience of Gattegno (1964-66)'s Silent Way of teaching a foreign language. This method involves the teacher working with a group of students intensively - without speaking in the initial stages. Alf had been to an Italian weekend course and Laurinda to a brief Spanish session followed later by a Japanese weekend course.

Alf realised that he had introduced a number of concepts to classes, after the 'Silent Way' weekend, by intentionally writing on the board in silence.

4) Working in the school - silence in year 7



Alf 20/09/95: We were going to do some work on arithmogons - I drew one, put a number in two circles, paused and filled the box in in between by adding, pause ... put a number in the third circle ... filled in the other two boxes by adding in the same way. A few hands had gone up; there was silence. Another example, still silence, a couple of students bursting to tell the answers in the boxes, but still silence (this was surprising). I was making eye contact with many of the class and looking a lot at a girl who I felt might be the last to pick up what was going on - there was concentration, but still no understanding on her face. A third example ... I turned to look at the class and *everyone's* eyes were burning into the board - I hadn't experienced this before and I almost broke down. Still silence ... I now filled in two boxes with answers from the class, everyone's hand seemed to be up except the girl; she was straining and seemed to have just understood; she half whispered an answer, not quite committing herself - but it was correct. One more example ... two boys had lost concentration, staring brought them back. The girl's hand was now up with the rest - the boys seemed to be following, so I nodded at her and a correct answer came. I then drew an arithmogon with only the boxes filled in and invited the class to try to find what the numbers in the circles could have been ... no one needed a further explanation, which is a rare event for me!



Laurinda: From my lesson observation notebook 18/09/95 (see Commentary 4 for explanation!)

- Strategies for 'knowing that they *all* know' (first bit) ... wonderful build of energy here! - a sign to move on often and articulation can get in the way ... a move direct to the filled in squares (done/said before your decision) ... (yes ... that's what you did).

Commentary 4

The story of the lesson illustrates an example of Alf experiencing a change in perceived behaviour of the students in relation to his changed behaviour. The energy of the students in response to his waiting for more of them to offer was what had been surprising.

The lesson observation notes need some expansion, however, because in this case there is clear evidence of Laurinda also learning. The first phrase (a macro-strategy or purpose) refers to the fact that Alf was demonstrating a technique (micro-strategy) of waiting for responses which had the effect of allowing enough time for more students to be able to offer. There was a wonderful build of energy into the completion of the second arithmogon. However, at this stage Laurinda felt uncomfortable since she was aware that she would have made the decision to move on to the next stage of presentation of the challenge of 'what's in the circles given the filled in squares'

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The rather strange, hesitantly offered, next phrases were interpreted in conversation after the event as evidence for this discomfort. Laurinda felt that either the energy in the students might become dissipated by the task being perceived as too easy or continue to build and need release in mayhem! Neither of these positions held and all children in this mixed ability group became completely absorbed by the fourth example. The energy levels of the students were high and the final two statements in the notes confirmed Alf's move to the challenging problem. This incident continues to provide us with questions about the links between energy, motivation and the use of silence in the mathematics classroom.

Within the theoretical perspective this exploration of the use of teacher silence (the macro-strategy or purpose) was here developed through micro-strategies of (1) giving the students a visual task offered slowly and silently which focused the attention of the students and (2) timing a change from do-able to more challenging yet related task. Having experienced the power of this technique under these circumstances both Laurinda and Alf were in the position of having implicit beliefs or central strategies brought into question having experienced a learning episode which they valued for the students. What were these central strategies? Further discussion would suggest that 'autonomy' was an inhibiting factor for Alf in the first year of teaching which could now be adapted as a theory given his increasing ability to take the authority position. For Laurinda it was to see that it was possible to continue the energy build using silence without boring the brighter students for the sake of the 'girl'. In this case the evidence was that the students were not bored and attacked the challenging problem.

5) Silence and energy versus stillness - a new common frame

A year 7 (ages 11 - 12) class had been working at seeing whether there was a link between the rules to describe a function and the graphs of the function articulated, as a purpose, by Alf as whether they could know what the graph would look like without needing to plot the points. The class had been working in the first quadrant only. To introduce these ideas the class had been playing the function game (to be played as part of the presentation at PME20, Banwell et al, 1972). The game is often played in silence which focuses individual attention and energy in the moment on the task of sorting out what is going on.

In discussions before the lesson the decision had been made for Laurinda to begin by inviting the group to share with her what they had been doing with Alf in the previous lesson. After negative numbers had been introduced into the game Alf would then refocus the group's attention on the purpose from the previous lesson and extend the work into plotting graphs in all four quadrants. At the handover Laurinda had used the micro-strategy of holding the silence longer than she would normally have done to force every student to commit themselves to an answer which in this case was negative. As the pens passed to Alf the students were excited and present in the task, yet not silent nor easy to handle! It was essential that Alf provided the move from this doable to a more complex yet related challenge so that the energy could be used.

Into an energised space Alf said 'What's different about what you've been doing with Miss Brown and what we were doing last lesson?' The effect of this question was to make the group absolutely still. This was a silence in the students which was of a different quality to that experienced before - a stillness.

Commentary 5

The silence energy which we had identified in the first visit described above was energetic and mobilising for the learners, their attention was focused in the present and they were using themselves in the moment bringing everything they have with them to restructure their experience. The contemplation (Gattegno, 1987) stillness seemed to be trapped by the invitation to compare two aspects of experience, before and after and it was as if their presences went away to attend to that difference as they looked inward. This stillness was also powerful but not about experience, more about integration (Gattegno, 1987). There had been the question and this was the reaction - this stillness was their will, not imposed by the teacher.

Some final thoughts and directions for future work

We work within the realm of what Bruner (1990) calls a 'culturally sensitive psychology':

> '(which) is and must be based not only upon what people actually do but what they say they do and what they say caused them to do what they did. It is also concerned with what people say others did and why ... how curious that there are so few studies that (ask): how does what one does reveal what one thinks and believes (p16-17).

The interaction of the two researchers/teachers also works creatively as ideas coemerge and implicit theories of learning and teaching were made conscious allowing the possibilities of adaption. There is now not so much of a feeling of an unbridgeable gap for Alf but a sense of staying with the uncertainty and developing ways of working on his practice:

> 'One discipline that has come out of the work is that of 'staying with the story'. In my notes on teaching in the first year, the observations are in general distant - about whole classes - with observation and analysis all mixed in ... forcing myself to hold back the analysis and stay just with stories about individuals or groups the analysis from this data then has the possibility of throwing up something I had not been aware of before.' (Alf, 3/12/95)

There was also evidence here for the students being engaged in and enjoying the challenge of mathematics. Here there is 'silence on the part of the teacher(s), so that they can clearly hear the verbal messages of the students' (Gattegno, 1971) and support for a listening classroom (Davis 1995). The voice of the students, repressed in the delivery mode of teaching where they listen in silence to explanations, is found in the task of interpreting ever more complex challenges.

We agree with Edwards and Núñez (1995) that the students' understandings be seen as 'appropriate, adaptive expectations arising from their own embodied experience in the world' and that the teacher 'must create situations in which the new mathematics is more adaptive than the old'. Our current work is developing our practical skills in sharing purposes and responses in the classroom so 'teaching with an emphasis on communication' (ibid, 1995)

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Within this realm we seem to be developing an image of whole class teaching which is neither exposition and practice nor individualised scheme but allows coemergence (Reid, 1995) of mathematics within the classroom and a tentative thesis would be that purposes act as mechanisms for the students and teachers to 'stay with the complexity' within the classroom thereby accessing their intuition (Gattegno, 1987).

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THE CULTURAL EVOLUTION OF MATHEMATICS

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The ever increasing rate of social evolution results in the demands being placed on mathematics becoming ever more complex. This paper considers the effect of mathematics being stretched between the needs of increasingly diverse subcultures whilst the rapidity of change causes a re-evaluation of traditional roles such as those between teacher-pupil and parent-child.

As teachers, we rightly value the way in which our students bring meaning to the mathematical situations they encounter. It seems there is much scope for judgment, insight and creativity in the style of mathematical work being introduced in many schools and we may aspire to encourage these qualities in the children's learning. Yet there is still a need for an individual to reconcile his or her own personal mathematical understanding with the ideas and traditions growing out of centuries of mathematical exploration and invention (cf. Ball, 1993; Brown, 1994 b). It is all very well students being creative mathematicians but they still need to be able to do everyday calculations and understand aspects of conventional mathematical thinking. We are often torn between attempting to focus on our student's own way of seeing their mathematical endeavours and seeing these endeavours with our own eyes, inspired, maybe, by a "correct" view of mathematics. There are, inevitably, difficulties for us in making sense of student's own developing understanding without using our own "expert" overview as a yardstick, especially when we pose the tasks they are working on. Teacher descriptions of children's learning often presuppose an adult overlay framing the mathematical ideas supposedly being addressed. Meanwhile, the sort of constructions students are likely to generate for themselves are a function of their own particular concerns in relation to the sorts of tasks they are presented with. Whilst we may wish to encourage students to pursue their own mathematical concerns at times, we retain the option of blowing the whistle and denying that their work is mathematics at all. Furthermore, as the apparent relevances of different aspects of mathematics, as perceived by society, grow or fade, the nature of tasks on offer and the values associated with them will alter. The ground does not rest beneath that which is called "mathematics". This paper considers how social evolution results in a fragmentation of any holistic notion of mathematics and forces a reevaluation of the teacher-student relationship

Taking a broader view, any society can itself be seen as being shaped as an idea by people who tell "stories" about it. There is, necessarily, a self-reflexivity to the very notion of "society". Mathematics, likewise, can be seen as being conditioned by its position and shaping within such stories and this produces a multiplicity of co-existing views regarding what mathematics is and the role it serves. The creation of these different camps can result in conflicting demands being placed on it. A particular dispute emerges around attempts to decide whether mathematics is in the world or in the mind. The disparity between these views is not resolved through a closer examination of what mathematics "is". One can, however, look at the social space which gives rise to these different perspectives and seek to understand the processes at work. But even within this task one needs to be alert to potential disputes regarding the way in which an independently existing mathematics might be invoked to arbitrate between the perspectives being examined.

As an individual I declare my own identity through asserting my identifications with various groups, by participating within them. Through doing this, both the group and I, myself, evolve (Habermas, 1991). Habermas' broader project focuses on what it means to create commonality and communicative links between different forms of life (see White, 1988, p. 154). He sees the building of these links as an integral element of growth, where both social and individual evolution are bound up with attempts to reconcile social practices with descriptive practices (Habermas, 1987, p. 60). Such practices however, can be highly localised and for this reason he seeks to develop a multi-dimensional concept of the world, as experienced by its various inhabitants, towards integrating alternative practices.

To engage in mathematical learning, not only is it necessary to share understandings with other individuals, one also needs to be able to participate in and move between a variety of mathematical subcultures; "everyday" mathematics, various types of school mathematics, examination mathematics, novice vs. expert mathematics, etc. each with their own particular language, scope of interest, values and associated skills (cf. Cobb, 1994; Brown, In press; Lave, 1988; Carraher, Carraher and Schliemann, 1985). Insofar as mathematics is socially constructed there is a need to examine the way in which the particular subculture flavours the mathematics it uses and how these various demands made on mathematics evolve as social needs change. As an example, the variation in styles of questions employed by different examination boards within the United Kingdom, results in students not being examined in mathematics per se, but rather, in the particular style of questioning the board chooses to offer. Similarly, the "geometry" I did in school twenty five years ago with protractors and set squares is qualitatively different to the "geometry" present in work with Cabri software (cf. Pimm, 1995; Laborde, 1993). I claim my own school's penchant for wobbly compasses had considerable effect within my formative experiences of what constituted a circle!! Classroom technology both encourages and facilitates a shift of focus. The mathematics itself is different, not just its presentation. Old points of anchorage get eased out. For example, a triangle in Cabri environment is qualitatively different to one in a pencil set square and compass environment, or one made with plastic strips, or one created on a computer using BASIC graphics or LOGO. In each situation the triangle appears in an environment where certain things can be done to it; it is a function both of available operations and of things that can go wrong. Each environment hosts a particular style of work or mode of culturality. The triangle appears in the specific cultural discourse, differentiated from the things around it, in terms of both its innate qualities and the way it is produced in the particular field. David Pimm (1995, pp. 56-58), in homage to Magritte's painting of a pipe known as "Ceci n'est pas une pipe", speculates:

"This is not a triangle".

Magritte's joke however does not transfer successfully. For it to work we would need to assume the primacy of a transcendentally defined mathematics, existing outside most cultural practices. In many cultural discourses a pencil drawn triangle is indeed a triangle, not just "as good as the thing itself for our purposes" (ibid.), not merely a nice drawing that helps us see the thing itself.

Mathematics in learning situations is generally subject to such cultural and stylistic flavouring. This sort of concern invites a shift from having an overview of mathematics *qua* mathematics, towards understanding how it is to engage in particular versions of it, within given social settings (cf. Mellin Olsen, 1987, pp. 18-76). Further, there is a need to understand how children participate, with their teachers, in the constitution of classroom mathematical practices (cf. Cobb, 1994, p. 15). Habermas' project is concerned with breaking the shackles of conservativism (see Huspek, 1991) and seeks to prevent this constitution from being mere reproduction. Participants in mathematics lessons, for example, would need to build a clearer understanding of how their actions relate to norms inherent in the particular subculture and how the criteria might change as they move into a new subculture. That is, students need to become more aware of the parameters of their own learning, towards being able to take a critical stance of how these parameters govern their situation. Skovsmose (1994) advocates an increased emphasis on thematic project work within mathematical learning to enhance student awareness of how problems may be contextualised. Meanwhile, Mason's work focuses more closely on the parameters of mathematical problems themselves (op cit.). The insider perspective needs to be understood more closely in relation to the contextual forces operating on it. An essential aspect of this that needs to be addressed by empirical mathematics education research is an analysis of how the symbolic framework employed within a given subculture mediates access to the understandings of that culture.

Like Cobb (1994), I feel it is inappropriate to insist that either individual or social perspective takes precedence. In looking around me I have some awareness of the conventions inherent in my culture and how they influence the way I see things, quite independent of any explicit educational programme alerting my attention to this. As an individual I have some concern with how I fit in. This seems inevitable in a world where so many subcultures confront each other and maybe some offer me membership. However, in line with Habermas, I suggest the task of education must be, in part, concerned with enabling the student to develop this self-reflective critical stance in relation to the perspective he or she assumes. More than ever, students are preparing themselves for a world undergoing fundamental structural changes, comprising rapid growth and increasing diversity. This creates pressure to regenerate styles of teaching and learning mathematics whilst moving away from treating mathematics as though it could be conceptualised in a stable state in relation to the reality it serves (cf. Brookes, 1993, 1994).

There are, however, a few remaining difficulties, which make the policy implications of such a view less than clear. Initiation will always remain an indispensable dimension of mathematical activity. Indeed it would be hard to conceive of mathematics teaching without initiation; the whole enterprise would dissolve in to thin air. We need to set something up before we can engage in a critical attitude. Nevertheless, in an environment of rapid environmental change we are increasingly concerned with enabling children to respond positively to constant renewal. The old ways cannot be our only concern. Indeed as adults we can only assume partial control in making the decisions. Teachers and pupils invariably share responsibility for structuring the space they share but as my colleague Olwen McNamara has pointed out, adults often find themselves very much at the beck and call of their offspring in such shared space. The child's action activates a response on the part of the adult trying to cope; the adult's behaviour forming itself around the child's whims. (Apologies to my baby son Elliot!). But things need to be in motion or, at least, anticipated, before resistance is possible by either adult or child.

I return to David Pimm's intriguing account, where he examines some of the issues associated with a move towards a more critical mathematics education. He offers examples of a number of questions which could only arise in a discourse designed specifically for the teaching of mathematics. The stories are daft in the sense they would not have any function outside of a pedagogic frame. (E.g. If one Confederate soldier kills 90 Yankees, how may can 10 Confederate soldiers kill? (Pimm, 1995, p. 64)) But what are the more honest alternatives? To engage in critical mathematics education it is a prerequisite that we embed the mathematics in a social practice, even if this is simply the social practice of doing school mathematics, but all too often the attempt to embed mathematics results in a curious world that exists only for mathematics classroom. On the other hand firm emphasis on a more pure mathematics, situated only in the practice of doing mathematics, can result in an alienating discipline restricting access to all but a few and concealing its applications in broader social practices. Pimm (Ibid., pp. 153-158) questions the virtue of giving primacy to critique in the way Skovsmose seems to. There is a risk that if children focus too much on critique they do not learn mathematics. Skovsmose's approach which emphasises teaching mathematics within social themes, certainly would reorientate the way in which we value mathematical concepts, placing more stress on those with practical potential, but then which practice? We risk introducing all sorts of artificial worlds as a proxy for social realities. However, if we reject such a move we are still left with deciding what it is we do promote. Do we make some serious investment in embedding mathematics in order to develop skills of critique or concentrate on going with some version of mathematics itself? John Redwood, a right wing government minister, recently questioned the value of comparative religion in British schools on the grounds it meant that each religion examined received no more than lip service. To embrace a religion you need to make a total commitment, dwell in it as an act of faith. He argued that children are not equipped to hold on to too many versions simultaneously and they should instead focus on one. But, of course, such a policy results in one religion asserting its power over all others. Similarly, if we invest too much faith in one version of mathematics we risk forcing students in to conservative reproduction of the status quo and providing short cuts for getting there. However, we are unlikely to face such an extreme polarity of choice.

At some point we need to concern ourselves with the issue of how far the teacher can take responsibility for facilitating initiation, in relation to the students building significance for themselves (cf. Brown, 1996) but, before this, we still need to decide what it is we are initiating students in to; various cultural practices, some sort of disembodied mathematics or, more probably, somewhere between. We are necessarily in a state of moving between situated mathematics, of which stories can be told, and a mathematics understood as if existing outside of everyday human discourse or, at least, functioning as a discourse with a very specific structure. Whilst the stories that Pimm cites lack credibility in the real world, all such stories would to a degree. They more or less embody some view of abstract mathematics and we can never have anything more. We need such stories to mediate experience, to help associate sense and reference. "Maths-speak" is, from a post-structuralist perspective, just as much a story as all the other stories, although some stories may appear more overtly mathematical than others and so more transparently reflect the status quo? Our overarching task is to nurture a dialectic between reality and stories told about it, a dialectic which transforms both. As Pimm suggests, we need to combine descriptive with generative language and, I suggest, not just generative in the existing frame. Stories cannot usefully function as a mere map but also need to trigger renewal.

In summary, there are many versions of mathematics engendered through practice in a multitude of subcultures. For a teacher there is always a decision as to whether mathematics should be situated in "practical" examples so as to make it more meaningful and accessible. However, many attempts at situating mathematics result in an artificial world created purely for pedagogic purposes. Meanwhile, the teacher may or may not wish to make initiation of their students into the status quo their principal concern. Whatever they decide it may be that students construct their own agenda anyway. Combining the teachers perceived intention with an outside view on how this intention is associated with significance to the student, we may begin to speculate on how much a teacher's input ensnares the student into reproducing the status quo and whether this is desirable or not. The task of the teacher seems to always involve initiation to a degree but if too many short cuts are taken in effecting this initiation, learning becomes compliance, resulting in students being ill equipped for facing new situations. We also risk facing a broader scale version of the Brousseau's "didactical trap", namely, the more we seek to specify the nature of a more pure mathematics the more risk there is of students not experiencing it. Teachers, however, need to find ways of enabling students to engage mathematically beyond the frames teachers themselves offer.

So then, I am floating the idea that mathematics is located in the practice of engaging in mathematical activity (cf. Brown, 1991, 1994 a). Mathematicians learn their skills whilst engaged in such practice. And that mathematics has a human dimension both in its historical creation and in its direct interface with individual humans. Mathematics is not so much oriented around universal facts but rather revolves around personal awarenesses engendered through practice. In learning mathematics a human is concerned with how he or she may use mathematical ideas and with his or her performance in mathematical activity where mathematical knowing involves some self-reflection on that known by the learner. The discipline of mathematics per se is not sufficient in itself in informing action; there needs also to be some development of practical skill of engaging in mathematical activity as understood in broader social practices. For this we need to know a little about the reflective state of being engaged in a mathematical task. Communication involving teacher and student is, at least in part, about operating on knowing through helping students to become sensitised in mathematical situations. This knowing, however, emerges through action since the social practices which host specific actions are imbued with the society's preferred ways of seeing things. Indeed, in this respect the individual cannot be seen as separate to his or her society nor the mathematics it uses - since the society *speaks* itself through the words and actions of its individual subjects.

Choices for teachers are not clear. Conceptions of mathematics change variously in a multitude of practices whilst governmental preferences in curriculum design fluctuate in response to rather different agendas. Criteria for professional advancement are not always commensurate with meeting perceived local needs. Teachers, however, must assume some sort of professional identity if they are to build up resistances to increasing environmental pressures. The teaching environment has become too complex for all demands to be complied with and so teachers are forced into making choices, both constrained and enabled by the structural framework they meet. Rapid change seems to have become a permanent condition rather than being a mere short term phenomenon seeing us through to a new stable state.

These demands have been addressed in some quarters. For example, Brookes has considered issues in education resulting from rapid environmental change.

> ...practice is governed by belief and that in changing circumstances there is a need to work at those implicit presuppositions which prevent accommodation to new conditions.
> The co-existence of an increasing variety of possibility in the world with the constant universal commonness of growth from birth to maturity implies unavoidable conflict.

-When the rate of change of environment is faster than the rate of human generation there is an implicit destabilising.
-This is seen as a constant state of resolving; that is of control in changing from state to state. It implies dynamic stability; *homeorhesis (stability of flow) instead of homeostasis (stability of state) (Brookes,* 1986)

These principles locate a desire to address the failures of teaching, of learning and of mathematics to keep pace with demands placed on them. Our attempts to describe them always lag behind their reality. Brookes (1994) offers the example of "computer studies", as taught in British schools until recently, where "environmental" change of computer knowledge and skills was far greater than the response time needed for the necessary change in the exam system. It was supposedly a "modern" subject but permanently out of date. Mathematics like any curriculum subject is, to a large measure, a function of the demands placed on it. As these demands evolve the subject itself comes under increasing pressure to change. What appeared static begins to strain as its pace of evolution overtakes that of generational change (i.e. pertaining to changes consequential to normal biological growth). Brookes (ibid.) argues that the recent penchant for criterion referenced testing in curriculum documents within the United Kingdom is "an intellectual product of engagement with a strongly behaviourist form of thinking involving a belief in identifiable aims and objectives for any particular educational enterprise". Such a move, he sees as an attempt to force adaption to a new order, sterilised in the language of the old. Brookes continues: "The recognition that the world is changing rapidly casts doubt on any programme which depends on rigidly defined propositions embodied in a static educational theory not capable of responding to environmental change". British teachers have recently experienced a succession of such curriculums, with each new set of guidelines modifying the last to a significant degree. The teachers did not have the opportunity to live in a particular version for more than a few months before yet another change was demanded. The disorientation and fatigue brought about in teachers by such rapid change resulted in a promise from the government not to change anything else for another five years to appease a disillusioned teaching force. Brookes (ibid.) suggests that for education systems to be compatible with the world as we experience it we need to "accept the twin constraints of an environmental framework that is changing non-repetitively and accelerating and a generational framework which is cyclically repeatable and only gently changing". The rapidity of current environmental change is such that educational policy grounded in the beliefs of the last generation and how they did things, loses touch with the challenges faced by the new generation. The very relationship between successive generations is challenged through past roles, assumed between old and young, father and son, teacher and pupil, being undermined. To meet the demands of such growing pressures students should be equipped to generate and work with their own accounts of the realities they face rather than rely too heavily on the accounts provided by their elders. The structures which govern our actions in mathematics education, however, are embedded in the very language we use; our mathematical language, the way governments describe mathematics on a curriculum, the way we describe our individual intentions as a teacher, etc. Increasingly, none of these ways of talking exist for very long and often become invalid before they become familiar. Their chances of surviving and being passed on to the next generation become ever smaller.

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BEGINNING LEARNING NEGATIVE NUMBERS

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In this work we put forward a way of learning negative numbers and also give the main conclusions we have reached from our research in this area. The basic characteristics of our proposal are: a) consideration of the extension of non-negative real numbers to real numbers; b) attention to three dimensions of numerical knowledge, that is, abstract, contextual and the number line; c) basing learning on a meaningful and concrete process through problem solving.

INTRODUCTION

Research into the learning of negative numbers has focused on various aspects. Some authors have paid attention to the moment in the learning process when these numbers can be introduced and to the students'previous concepts about integer arithmetic (Murray, 1985; Human and Murray, 1987; Davidson, 1987). Other researchers have focused on the role of the number line (Küchemann, 1981; Peled, 1991) or on the solving of additive problems (Vergnaud, 1982).

The teaching of numbers spans a large period in students'school life. Children begin by studying non-negative integer numbers Z_+ and end with real numbers R, passing through several intermediate stages. We believe that the most delicate moments in this long process are precisely those when extensions are made from a numerical set to another, when it is necessary to relate the new numbers, the operations performed with them and the order between them with those numbers already known. As such, numerical extension should not mean a break in numerical knowledge but rather an extension in knowledge by integrating old and new knowledge in a single whole. The purpose of the present work is to put forward a proposal about how to achieve extension to negative numbers. The basic ideas about how this can be done can be summarized as follows:

1) We consider the introduction of negative numbers starting from non-negative real numbers R_+ and so we study the extension $R_+ \to R$. In some countries, Spain for example, the extension studied in schools is $Z_+ \to Z$.

2) Knowledge of negative numbers is manifested in several dimensions (Peled, 1991). Pure and symbolic mathematical knowledge is found in the *abstract* dimension. Use of numerical knowledge in concrete situations is found in the *contextual* dimension. Finally, identification of numbers with points on the number line is found in the *number line* dimension.

3) Learning is carried out by solving several types of problems in different contexts and with different structures. We have studied the identification of addition and subtraction, as this is a basic idea

In this work we also present the main results we have obtained from a classroom experiment based on our proposal. We worked with three experimental groups (G1, G2 and G3) and two control groups (G4) of 12-13-years-old students in three different schools in Tenerife (Spain). We had prepared curricular material which was used by the students for approximately two months (4 or 5 hours per week). In the experiment the students carried out various tests and 11 students (S1, S2, ..., S11) belonging to groups G1, G2 and G3 were selected to undergo clinic interviews. These students were chosen on the basis of the different levels of knowledge shown in the tests they carried out.

EXTENSION TO REAL NUMBERS

There are several ways of extending Z_+ to arrive at R. For example, in Spain, extensions are carried out in the following sequence:

$$Z_+ \longrightarrow Q_+ \;; \qquad Z_+ \longrightarrow Z \;; \qquad Q_+ \longrightarrow Q \longrightarrow R.$$

The lack of continuity in the above sequence is obvious. We believe that a suitable learning sequence is the following:

$$Z_+ \longrightarrow Q_+ \longrightarrow R_+ \longrightarrow R \;,$$

whereby there is a continuous progression towards R. One of the aims of our experiment was to check that this sequence of extensions does not mean a poorer understanding of Z. Thus, groups G1 and G2 did the extensions $Q_+ \rightarrow R_+ \rightarrow R$, although the main work was focused on the extension: $R_+ \rightarrow R$. Group G3 worked on the extension $Z_+ \rightarrow Z$, with the same material as groups G1 and G2, but using integer numbers only. The control groups (G4) worked on the extension $Z_+ \rightarrow Z$ using their textbook, except for some changes needed to make contrasts with the experimental groups.

In other research (Davidson, 1987) the possibility of introducing negative numbers at early ages has been studied, allowing a sequence like the following to be carried out:

$$Z_+ \longrightarrow Z \longrightarrow Q \longrightarrow R$$

The results of the tests undertaken by the students lead us to believe that working with extension $R_+ \rightarrow R$, instead of with $Z_+ \rightarrow Z$, does not mean poorer knowledge of Z. In order to illustrate this conclusion, we present in Table 1 average results $\mathbf{x}$ ($0 \leq \mathbf{x} \leq 10$) and the standard deviation $\mathbf{s}$, for groups of questions on integer numbers: order, addition-subtraction, multiplication-division, problem solving and problem writing (giving the students an operation so that they write a problem).

The data in Table 1 show similar knowledge in all groups regarding the different aspects of integer numbers. In other words, the groups that carried out the extension to R did not achieve an inferior level of knowledge of integer numbers.

Table 1. Results of questions about Z

	G1	G2	G3	G4
Order	$x = 8.2$ $s = 2$	$x = 8.6$ $s = 2.3$	$x = 8.3$ $s = 2.2$	$x = 8.4$ $s = 2.4$
Addition Subtraction	$x = 8.6$ $s = 1.6$	$x = 8$ $s = 2.5$	$x = 7$ $s = 2.5$	$x = 7.5$ $s = 2.3$
Multiplication Division	$x = 8.4$ $s = 1.5$	$x = 7.4$ $s = 2.8$	$x = 6.8$ $s = 2.7$	$x = 7.6$ $s = 1.9$
Problem solving	$x = 7.3$ $s = 1.8$	$x = 6.6$ $s = 1.9$	$x = 5.6$ $s = 2.2$	$x = 6.4$ $s = 2.3$
Problem writing	$x = 8.5$ $s = 1.8$	$x = 6.6$ $s = 1.9$	$x = 5.6$ $s = 2.2$	$x = 6.4$ $s = 2.3$

THREE DIMENSIONS OF KNOWLEDGE

Peled (1991) concluded that the descriptions made by students about how they perceive negative numbers and operations using these numbers can be considered in two dimensions of knowledge: a quantitative dimension and a number line dimension. Our research leads us to divide the quantitative dimension into two, which we call the abstract dimension and contextual dimension.

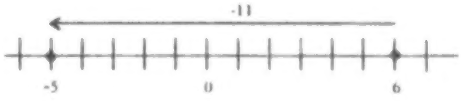
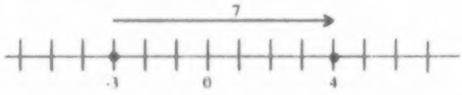
Abstract dimension. This refers to the abstract and symbolic knowledge of numbers, order and operations. For example, an aspect of this dimension is the addition $3+(-8)=-5$ obtained when applying a rule of type "to add a positive number and negative ones, subtract absolute values and write the sign of the greater value".

Contextual dimension. This covers the knowledge expressed in concrete numerical situations. For example, associating the number -2 with "I owe 2" or "2 below". In our research we have worked with numerical situations in six contexts: have-owe, temperature, sea level, road, time and elevator. Furthermore, the numerical situations considered correspond to states ("I owe 2"), variations ("I lose 2") and comparisons ("I have 2 fewer than you").

Number line dimension. This dimension is seen when making representations of numbers, operations and order on the number line.

The level of knowledge in each dimension for negative numbers is different for each child (Peled, 1991). We have studied the transferences of knowledge made between the different dimensions. So, in the interviews made with the 11 students, we posed questions in which they were asked to pass the knowledge given in one dimension to another dimension. In total, we posed 18 to 20 questions for each transference. In Table 2 some examples of the questions made to the students are given.

Table 2. Example questions about transferences between dimensions

abstract → number line Represent on the number line the operation -4+10=6	**number line → abstract** Tell me an operation that might be represented on the number line in the following way: 
abstract → contextual Tell me a situation that might be solved using the following operation: -4+7=3	**contextual → abstract** Tell me an operation that might solve the following situation: *The temperature in Madrid is 9 degrees above zero. In Paris it is 12 degrees less than in Madrid. What is the temperature in temperatura in Paris?*
contextual → number line Represent the following situation on the number line *An elevator was on the floor 5 of the basement and went up 7 floors. After going up in this way, what floor was the elevetor on?*	**number line → contextual** Tell me a situation which might be represented in the following way: 

The percentages of transferences carried out correctly are given in Table 3.

Table 3. % of correct transferences between dimensions

	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11
students'levels	L	L	ML	ML	M	M	M	MH	H	H	H
abstract →number line	6	6	29	25	44	68	44	43	82	83	87
number line →abstract	6	18	25	37	50	75	50	56	84	81	93
abstract→contextual	31	31	31	43	61	43	37	62	72	62	77
contextual →abstract	55	42	55	47	80	63	55	77	95	94	83
contextual→number line	50	30	85	45	85	85	90	75	100	100	95
number line→contextual	44	40	72	55	90	75	89	95	100	95	100

Students'levels: **L** = Low, **ML** = Medium-Low, **M** = Medium, **MH** = Medium-High, **H** = High

Here, in brief, are some of our conclusions:

Asymmetry in transferences. In Table 3 it can be seen that the differences in transferences between the abstract and number line dimensions and between the contextual and number line dimensions are not excessively wide. However, there seems to be an asymmetry between the contextual and abstract dimensions. For all students, *it is more complicated to pass from the abstract to the contextual than from the contextual to the abstract.*

Difficulty in transferences. From Table 3 it can also be concluded that there are wide differences between students in a single transference, depending on the student's level (for example, from a 6% for student S1 to 93% for student S11 in the transference from the number line to the abstract). On the other hand, there are certain constans in students' answers:

It is easier for all students to arrive at the representation on the number line from the contextual than from the abstract.

It is easier for all students to arrive at the contextual from the number line than from the abstract.

It is easier for nearly all students (except S3 and S10) to arrive at the abstract from the contextual than from the number line.

Representation of states and variations. Students differentiate states and variations, representing them with points and arrows, respectively. Those students who cannot differentiate them are of a low or medium-low level. Bell (1986) noted that students have difficulties in handling combinations of transformations (addition of two transformations resulting in the total transformation, $t_1+t_2=t_t$) when they do not know the starting point. We have observed that in order to represent variations there are some students who need a point for the initial position (this need varies in accordance with students'levels).

Types of situations used. It can be seen that when students invent contextualized situations there are differences depending on the initial dimension. When they invent situations on the basis of a representation on the number line they use contexts (elevator, road, etc.) and structures that they do not use when the initial dimension is abstract, when they always use the owe-have context.

On the other hand, there are situations of symmetrical transference between the three dimensions, for example, a simple situation such as "the temperature is 8 degrees below zero". However, there are more complex situations, such as some of additives, in which the three dimensions are not related symmetrically.

Representation of numbers in isolation. When it came to representing on the number line an operation such as $-4+7$ (that is, when passing from the abstract to the number line dimension) an error committed by some students was to indicate the three numbers involved in the operation in an isolated way on the number line, without demonstrating that for them the number line is a model where this operation can be calculated in a meaningful way. This error has been noted in the work undertaken by Carr and Katterns (1984), but with respect to operations with positive numbers.

ADDITIVE PROBLEM SOLVING STRATEGIES

In our opinion, the main difficulty in learning negative numbers is that the old notions of adding and subtracting, which seem to be contradictory with positive numbers, are integrated here in a single idea of addition. Our proposal is to use a "double language": for example, "he lost 2" is the same as "he won -2". We use this double language when solving additive problems. Following the classification adopted by Vergnaud (1982), the problems we used had the following structures:

- addition of two states resulting in the total state: $s_1 + s_2 = s_t$
- addition of two transformations resulting in the total transformation $t_1 + t_2 = t_t$
- addition of an initial state and a transformation resulting in the final state $s_i + t = s_f$
- addition of a state and a comparison resulting in another state $s_1 + c = s_2$

In the interviews we gave the 11 students various additive problems with the structures and contexts already referred to. In this section we show some of the strategies used by the students to solve these problems. We shall comment on those strategies we consider most important, either because of the frequency with which they are used, or else because of their interest in allowing us to know students' understanding of the three dimensions. In order to solve problems, students usually formulated an operation and/or made a representation on the number line.

Looking for operations that match the results on the number line. They solved the problem on the number line and then attempted operations until they came up with an operation whose result matched the result on the number line. This was especially the case in the more difficult problems, for example, those problems in which the unknown datum was the first or second term in the following equations: $s_1+s_2=s_t$, $t_1+t_2=t_t$, $s_i+t=s_f$, $s_1+c=s_2$; in other words, problems we call unknown 2 and unknown 3. So, although a student might formulate an operation, this did not imply that this operation was meaningfully related to the text of the problem.

Falsifying results of operations. This phenomenon is related to the previous one and comes about when students know the result of the problem, because they have done it on the number line, then formulate an unsuitable operation for the problem, and yet write the solution previously obtained on the number line as the result of the operation and not the result obtained from the operation they have formulated.

Having erroneous operational rules. Having erroneous operational rules sometimes leads to making incorrect formulations as solutions to the problems. This type of error comes about through the following sequence of actions: students know beforehand the solution to the problem, as they have done it on the number line; they wish to formulate an operation whose result matches that obtained on the number line, but, because their operational rules are incorrects, they formulate incorrect operations or operations unrelated to the problem.

Changing the structure of problems. Here students change the structure of the problem to another structure which is the consequence of the structure given and which is easier for them to understand. Usually this occurs with problems of the types unknown 2 and unknown 3. In the example given below, which is a problem with the structure $s_i + t = s_f$, where the transformation (t) is the unknown, some students changed it into a problem with the structure $t_1 + t_2 = t_t$, where t_t is the unknown. Example:

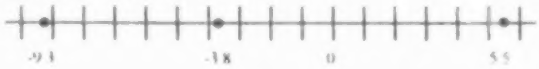
T: A child begins a game with *6 pesetas and finishes the game owing 5 pesetas. What has happened during the game?*
S10. *He has lost 11. Because at the end he owed 5 and at the beginning he had 6, meaning 6 + 5 = 11.*
-5, which was what was left at the end, -5-6 = -11. Because we take it that as he was left with less money than he had at the beginning, he had to lose the 6 pesetas, and he lost the 6 pesetas, and he lost another 5, which was what he owed, so he lost 11.

Interpreting results incorrectly. Sometimes the explanations given by the children reveal a faulty relationship between the contextual and the abstract, while at other times they seem forced justifications. For example, a problem where the result is the age of someone who lived before Christ can be justified by saying "this person's age is –20 because he lived before Christ".

Following the order and signs of the data in the text. A frequent way of solving problems was to formulate an operation with the numbers following the same order as stated in the text of the problem and with the signs showing the situations. The following example demonstrates this type of error.
T. A person was born in the year 15 before Christ and died in the year 7 before Christ. How may years did he live?
SI. -15 - 7 = -22
He was born in the year 15 before Christ, which would be negative, and he died in the year 7 before Christ, which would be negative. As minus and minus are added and the sign of the greater value is written. How many years did he live?...He lived 22 years.

Representing numbers in isolation on the number line. At certain moments some students fail to grant any meaning to the representation on the number line. So, for example, they place the three numbers involved in the problem as points on the number line without any relationship between them. In other words, they do not relate the representation on the number line with the problematic situation. This can be seen in the following example.

T. *The temperature in Madrid is 5.5 degrees above zero, and in Paris it is 9.3 degrees less than in Madrid. What is the temparature in Paris?*
S1. *Subtract and there are...*



This type of representation shows that the students is not granting any meaning to the situation. However, this is not because the situation involves negative numbers, as, in initial tests where students were asked to represent similar situations with positive numbers, this type of representation was also used (Bruno and Martinón, 1994).

CONCLUSIONS

- We believe it possible to make the extension: $Z_+ \rightarrow Q_+ \rightarrow R_+ \rightarrow R$ without any negative effect on the knowledge the students should have about integer numbers.

- We have studied three dimensions of knowledge of negative numbers and we can see that transference from the abstract to the contextual dimension implies greater difficulty than transference from the contextual to the abstract. It is easier to arrive at representations on the number line from the contextual than from the abstract.

- One of the main points of discussion about the learning of negative numbers concerns the use or not of the number line. Some researchers have pointed out the difficulty of adding and subtracting on the number line both with positive numbers (Carr and Katterns, 1984; Ernest, 1985) as well with negative numbers (Küchemann, 1981). Küchemann states that he number line should be abandoned as a model for the teaching of subtraction of negative numbers. Our results show that there are difficulties, but they also demonstrate that it is easier to begin learning the number line through contextualized situations.

- There is a certain disconnection between solving problems using the number line and solving by formulating operations. It is easier and more secure to solve problems using the number line than by formulating operations. This is shown in those actions we have called *looking for the operation* and *falsifying the result*.

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CONCEPTS OF MULTIPLICATION OF CHILDREN AND TEACHERS IN MOZAMBIQUE

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1. INTRODUCTION

Since August 1991, I am working with a group of about forty students who are following a special Master's Degree course in Mathematics Education for Primary Schools. The students are all primary teachers with at least three years of experience in teaching. I am doing the subject of Psychology of Learning Mathematics with them. The work the students and I did together lead to some insight in problems in mathematics learning in primary school in Mozambique. I want to dwell on one of these problems in this paper.

During the first four years of the course, the students had to concentrate on observing children doing mathematics. Through practical tasks they learned more about the children's way of thinking and not less about their own way of reasoning when doing mathematics. One of the tasks in 1994 was to realize some work sessions with primary school pupils about multiplication. Much information came out and several questions arose, among which the problem I want to present here.

Although 3 x 4 and 4 x 3 give the same result, there are different ways of "reading" (or interpreting) the multiplication. For some people 3 x 4 means 4 + 4 + 4, because they think it is "three times four" and for others it means 3 + 3 + 3 + 3, because they think it is "three repeated four times". In several reports of the students about their work sessions we found that these different interpretations of a multiplication can lead to confusion in teaching and learning basic facts of multiplication.

I present some short dialogues found in a report of student Z. She did work sessions about multiplication with André, a 13 year old pupil from grade 5. These dialogues illustrate clearly how the difference in the two ways of interpreting a multiplication may lead to confusion in teaching and learning.

2. DIALOGUES BETWEEN LEMEP-STUDENT Z. AND ANDRÉ, A THIRTEEN YEAR OLD PUPIL OF GRADE 5.

Student Z. did some tests about multiplication and found out that the pupil André had difficulties in memorizing the basic facts of multiplication when involving bigger numbers like 6, 7, 8 and 9. She decided to help André. In some work sessions she wanted to show him that there exist relations between basic facts. She argued that Andre will probably memorize the basic facts of multiplication more easily when understanding these relations.

2.1 Preparation of the work session

In her preparation for one of the work sessions with André, Z. decided to do the multiplications 6 x 9; 7 x 9; 8 x 9 and 9 x 9 with him. Her main idea in order to help the pupil was based on:

6 x 9 = 5 x 9 + 9
= 45 + 9
= 54

This idea is written in the Mozambican Mathematics schoolbooks for grade 3. (Draisma et al., 1984, p. 22-34). The book shows that there are relations between the basic facts of multiplication which can be useful for memorizing them and for building the times tables by adding always *once more the second factor*, for instance, 6 x 9 means 9 + 9 + 9 + 9 + 9 + 9, or 6 x 9 = 5 x 9 + 9. Building up a times table you get:

1 x 6 = 6
2 x 6 = 6 + 6
3 x 6 = 6 + 6 + 6
4 x 6 = 6 + 6 + 6 + 6 etc.

Note: After analyzing the work session Z.gave an important information: she said that she *used* to interpret the multiplication 6 x 8 as "six is repeated eight times" or, 6 x 8 = 6 + 6 + 6 + 6 + 6 + 6 + 6 + 6 , thus the *first factor is repeated.*, while organizing the times tables she *repeats the second factor:*

6 x 1 = 1 + 1 + 1 + 1 + 1 + 1
6 x 2 = 2 + 2 + 2 + 2 + 2 + 2
6 x 3 = 3 + 3 + 3 + 3 + 3 + 3

Constructing times tables, she adds *one unit more* to the second factor in order to get the next result. The three in 6 x 3 comes from 2 + 1 ("two units and one unit more"), so 6 x 3 = (2 + 1) + (2 + 1) + (2 + 1) + (2 + 1) + (2 + 1) + (2 + 1).

2. 2 What happened during the work session?

In several parts of Z's work session we found a mixture of two ways of interpretation of a multiplication, what leads to some confusion by André.

A. Student Z. wants to explain how the pupil can find the answer of 7 x 9 using the idea that 7 x 9 comes from 6 x 9 + 9, just as what she wrote in her preparation. (*Observation:* Earlier in the work session André found easily the result of 6 x 9 as 5 x 9 + 9.). Here follows a part of the dialogue:

Z: If you have got seven times nine (7 x 9), what are you going to do?

Here we have nine times six (indicates the product calculated before: 6 x 9 = 5 x 9 + 9 and the result)

A: ....(no answer)

Z: Now, seven times nine (7 x 9), what are you going to do?

A: ....(no answer)

Z: "Here we have nine times six and it is the same as six times nine "(and she writes 9 x 6 = 6 x 9) and asks again: "Seven times nine?"

A: "Is equal to nine times seven."

Comment

The first problem arises when Z. changes 6 x 9 into 9 x 6, without being aware the change she made. Later on it seems that she became aware of the change, when she said: "9 x 6 is the same as 6 x 9".

The point is, why did student Z. change from 6 x 9 to 9 x 6? Was it a mistake ? Probably, thinking about the multiplication 6 x 9, she thought "*six repeated* nine times" , because, as she said, she used to interpret a basic fact in this way.This interpretation leads to trouble. She started, showing the pupil that

6 x 9 means 5 x 9 + 9, and in this reasoning the *nine is repeated* and not the six. While changing 6 x 9 into 9 x 6, Z. wants André to find the relation between 9 x 6 and 7 x 9, but he didn't succeed.

Student Z. used two different interpretations of multiplication: for 9 x 6 she said: "nine repeated six times" and for 7 x 9 she said: "seven times nine."

In both tasks she is repeating the nine and therefore she thinks that her reasoning is as foreseen in her preparation and is done as in the schoolbooks. But in fact, she adopted two ways of interpretation of multiplication, which lead to some difficulties for the pupil: he couldn't give an answer.

B. In the following part of the work session another problem came out when Z. asked: "How many units is six more than five?.

Z: "We have nine times five and nine times six"."How many units is six more than five?"

A: "One unit"

Z: "You already did nine times five. How do you do with nine times six?"

A: "Do we add nine six times?"

Z: "Do you think we have to add all those times? If you already know nine times five, the nine is added how many times?"

A: "Five"

Z: "To get six how many is missing?"

A: "One time."

Z: "One time, of what?"

A: "One time six"

The pupil is reflecting, waits for a moment and then says: "One time nine".

Comment

The question "How many units is six more than five?" is originated by Z's way of organizing the times tables. For her the relation between 9 x 5 and 9 x 6 is that you add **one unit** to each 5 for obtaining the result of 9 x 6. But Z's main objective was to show the relation between the basic facts according to the interpretation of 6 x 9 = 5 x 9 + 9, as in the schoolbooks. In this case you add **one nine** more These two ways of building times tables are mixed up in Z's mind and lead to some difficulties by André.

— The answer "one unit" was easy to find for André. But he starts doubting when he has to give an answer to the question:"You did already nine times five. How do you do with nine times six?" In this question Z. wants to know which *factor has to be added once more..* In her mind the *"one unit more"* is used in order to get *"one time nine more"*. She thinks *"nine times six is nine times five and nine times one"*, or 9 x 6 = 9 x 5 + 9 x 1 and inverted the interpretation of "nine times one" into "one time nine more". But André learned and interprets
9 x 6 = 8 x 6 + 1 x 6. and therefore he starts doubting:

— We can observe that when he asks: "Do we add nine six times?"

— Later on, when the pupil has to answer of what has to be added once more, he first says "one time six", according to his interpretation of multiplication. But then he is reflecting, probably about what Z.had said earlier, "nine repeated six times", and then says "one time nine".

C. In the next part André is confused about which factor is repeated in the basic fact 6 x 8 and writes 6 x 8 = 5 x 8 + **6**. Z. is rather admired with the child's thought, not being aware that the child was influenced by her different interpretations of a basic fact of multiplication.

Z: Try to do 6 x 8

A: Writes: 6 x 8 = 5 x 8 = 40 + 6
= 46

Z: But, how? More six? That six, where did you find it?

A: I did five times eight.

Z: Five times eight. Yes, but this six (indicating the six in 40 + 6), how did it appear?

A: There were 6. Here I did five times eight, so then there is missing six more.

Z: How is there missing six more ? There were six and you took five of it. How much is missing?

A: One.

Z: One, of what?

A: .... (no answer)

3. CONCLUDING REMARKS

The report of this work session done by Z. shows us the following facts:

a) - Z. was used to interpret basic facts of multiplication in different ways and was not conscious of that fact.

b) - After analyzing the work she became aware that these two ways of interpretation can become a problem for teaching and learning when the teacher is not respecting the pupil's interpretation of a multiplication.

c) - There are different methods of building up times tables, what must be considered when teachers will show the relations between the basic facts.

d) - It is necessary to study the different ways of interpretation of a multiplication in the teacher training courses in Mozambique.

e) - The different interpretations are related to the language expressions used: looking at the written basic fact 3 x 4, some people will say "three times four"; others say "three four times"; others: "three repeated four times", or "three multiplied by four."

f) - It seems that the mother tongue used by teachers and children in Mozambique influences the way of interpreting a multiplication.

4. WHAT TO FIND OUT ?

Some research has started to look for the reasons of the use of two different ways of interpretation of a multiplication among Mozambican teachers and children. The research goes on and will focus on: a) the language used in multiplication;
b) the didactical aspects of teaching and learning multiplication; c) the question if there is any advantage in using only one interpretation of multiplication in lower primary school? If so, which interpretation should be chosen in Mozambique?

4.1 The language used in multiplication

a) In the LEMEP-course we have been studying both interpretations of a multiplication. Some students admitted that their interpretation of a multiplication could be influenced by their mother tongue, because in their mother tongue they will read 3 x 4 as "three repeated four times" Some of them argued that they have learned at school, where Portuguese was the language of instruction, that 3 x 4 means 3 + 3 + 3 + 3. Others said that they have learned at school that 3 x 4 means 4 + 4 + 4.

Jan Draisma, lecturer of Didactics of Mathematics in the LEMEP course, did a study about the expressions related to multiplication, used in the students' mother tongues. The results show that there is some interference of the Portuguese language into the mother tongue and vice versa (Jan Draisma, 1995).

b) During an excursion with the LEMEP-students to Zimbabwe, we saw several mathematics lessons in different grades at two primary schools. At the primary school of Chitungwiza, in a 3rd grade, the times tables of eight and nine were written on the blackboard in different ways:

on the left:	1 x 8	on the right:	9 x 1
	2 x 8		9 x 2
	3 x 8		9 x 3
	etc.		etc.

The LEMEP-student Mussa José, who attended the lesson, asked the teacher why she had written the times tables in different ways. She answered: "The pupils who speak fluently English prefer to use 1 x 8; 2 x 8; 3 x 8; and the pupils who are Shona speakers prefer to use 9 x 1; 9 x 2; 9 x 3." Unfortunately there was no time to discuss this aspect a little more with the teacher.

The point is, if you want to help a child to memorize the basic facts showing which relations exist between the basic facts, you must know how the child interprets a multiplication. And her interpretation is influenced by the language she uses. It seems the primary school teacher of Chitungwiza already understood this.

c) In "Teach yourself Swahili", under the title "How often?" the author states: "Note that, as **mara tatu** means three times, **"sita mara tatu"** means "6, three times" just the reverse of the English "six times three", although the result is the same. Anyone who has to teach arithmetic should make a special note of this, for the ignoring of this fact in the earlier stages is the cause of much of the haziness with which African children regard arithmetic." (D.V. Perrott, 1967)

These points are probably indications that the interpretation of a multiplication by Southern African, Bantu language speaking pupils and teachers, using Portuguese or English as medium of instruction, is influenced by their mother tongue.

4.2 The didactical aspects of learning and teaching multiplication

a) My first years of experience in education were in the Netherlands, where I worked for seven years as a primary school teacher. At that time I was not aware that multiplication could be difficult or a problem to teach or to learn. It was clear for me that 4 x 3 ("four times three") is equal to 3 + 3 + 3 + 3 and I was never doubting this meaning of multiplication. It seems to be a long tradition in the Netherlands that in a multiplication the second factor is repeated. (Fred Goffree, 1989; Adri Treffers and E. de Moor, 1990)

Acquiring new experiences in primary education in Mozambique, I was confronted with other ideas. I met many primary and secondary teachers who used to interpret 3 x 2 as "three repeated two times." So for them 3 x 2 means 3 + 3.

b) Some literature research was done on Portuguese books used for teacher training before independence in Angola and Mozambique. We found both ways of reading and interpreting multiplication:

— Didáctica das Lições de Aritmética da 1ª classe (2º ano) do Ensino Primário, (1964) This book was a guide for primary school teachers used in Angola and in Mozambique for many years before their independence came (1975). For introducing the meaning of multiplication two different expressions are used: a) "three twigs with two leaves each" b) "two leaves, three times, that makes six." (page 77 - 80)

— Elementos de Didáctica , Classe Pré -Primária"(1966).The author, Fausto Faria de Sá, was the teacher educator for didactics at the "Escola do Magistério Primário" (Primary Teacher Training College), in Maputo (formerly Lourenço Marques) for many years. He uses two ways of interpretation of a multiplication.The author is speaking about shoes and interprets 2 + 2 + 2 as 3 x 2. "*There are three times two shoes* ." In the same lesson he is speaking about cherries and he interprets 2 + 2 + 2 as 2 x 3. "*There are two cherries repeated three times* ." For both interpretations the author doesn't make difference in the picture: three groups of two, can be used for reading 3 x 2 and also for 2 x 3.

There are other Portuguese books on Didactics of Mathematics, that use only *one* interpretation. They are based on the following reading of the expression 3 x 4: "três vezes quatro" (= three times four).

- *Didáctica do Cálculo*, 1972; Gabriel Gonçalves, p. 180,

- *Didáctica Especial* , 1964; Francisco A.F. Queirós,

Page 27 e 28: Example: 1 + 1 + 1 + 1 = 4
4 times 1 = 4
4 x 1 = 4

The author of this book says:

> "The given example here about a multiplication in the quantity of four, poses a problem which we wish to explain, because *it goes against the traditional current of the indication of multiplication.*
>
> We put this operation in the line of an addition and this lead us to consider the multiplicand as a factor that is repeated and the multiplier as the indicative of the number of times that this factor is repeated. So therefore we say that:
>
> 5$00 + 5$00 + 5$00 = 15$00
> It is three times 5$00 = 15$00
> That is 3 x 5$00 = 15$00
> and **not**: 5$00 times three, or, 5$00 three times.
> because: 5$00 x 3 , or 5$00 3 x
>
> does not translate the logical sequence of the passage from addition to multiplication. The commutative property of multiplication has to be taught later and as we are in the beginning, (he means in the beginning of teaching children the multiplication) we don't see that we can do it in another way."

In several Portuguese schoolbooks for pupils, which were used in Mozambique for many years, 3 x 4 means 3 + 3 + 3 + 3 , because you say (think or read) "three repeated four times". But looking at the more recent mathematics books for primary school in Portugal, you will find that in many books a multiplication as 3 x 5 means 5 + 5 + 5. It seems that children in Portugal nowadays learn that in a multiplication the second factor is repeated.

c) In the Zimbabwean Syllabus for Mathematics at Primary School, of the Ministry of Education in Zimbabwe, 1987 (first edition 1984), children in grade 2 learn that for 2 (3) they have to say (or read) "two sets of three", what means 3 + 3, so *the three is repeated*.. But when they are in grade 3, they have to say (read) for 2 x 3 "two multiplied by three", and in this case 2 x 3 means 2 + 2 + 2; *now the two is repeated*..

It is evident that the two ways of interpretation of a multiplication by Mozambican teachers and children, is not only coming from mothertongue influence Several books, including books about didactics of mathematics, use two different expressions for reading a basic fact of multiplication. These books were guiding the teacher training courses in Mozambique during a long time.

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ON PRESERVICE TEACHERS' UNDERSTANDINGS OF DIVISION WITH REMAINDER

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Abstract

The focus of this study is on preservice teachers' understandings of numerical entities, expressions and terms pertaining to division with remainder. Many problems appear to result from the conflation of whole and rational numbers and inadequate understandings of alternative expressional forms of these entities. Data obtained from clinical interviews (n=21) indicate pervasive conceptual and referential difficulties with numerical entities and linguistic terms pertaining to them. Such difficulties were manifested in division tasks involving prime decomposition, calculators and the division algorithm. It appears that partitive dispositions towards division exacerbate many difficulties that quotitive dispositions towards whole number division with remainder may resolve.

Background

There is an increasing awareness and focus in research in mathematics education of the influence of informal knowledge and everyday experience that learners bring with them into the mathematics classroom (e.g., Mack, 1990). Furthermore, learners difficulties in relating practical, 'real-world' situations with arithmetic operations of multiplication and division has been a matter of intense and ongoing investigation (e.g., Greer, 1992). These difficult and important problems of informal knowledge and 'real-world' contextualizations with respect to prospective and preservice teachers' understandings of division have been the subject of intensive research in recent years (e.g., Graeber, Tirosh, & Glover, 1989; Ball, 1990; Simon, 1993), especially with respect to partitive and quotitive models of division proposed by Fischbein, Deri, Nello, & Marino (1985). A number of studies have also considered preservice teachers' subject matter knowledge in so-called 'decontextualized' problem situations involving divisibility (Zazkis & Campbell, in pressa), division by zero (Ball, 1990), long division and division with remainder using calculators (Simon, 1993). The objective of this study is to further investigate preservice teachers' subject matter knowledge of division with remainder in such 'decontextualized' tasks involving prime decomposition, division with calculators and the division algorithm.

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Theory

Terms such as 'decontextualized' or 'context-free' mathematical knowledge can be problematic if they are interpreted to mean independence from 'real-world' contexts rather than, as they are usually intended, independence from practical 'real-world' applications. It is important to qualify the former interpretation as it is highly questionable that 'abstract' understandings of mathematical entities and algorithms can ever be completely decontextualized or disembodied from 'concrete' real-world experience (Campbell & Dawson, 1995). Given that cognitive structure, and mathematical cognition in particular, emerges and co-evolves with such concrete experiences, the meaning of terms such as 'decontextualization' and 'context-free' cannot be taken to disclude embodied real-world experiences.

This is not to say that a distinction between concrete experience and abstract understanding is not a useful theoretical distinction to make. The question is where to draw the line. Here, the subset of 'real-world' experiences, sensorimotor and perceptual, involving symbolic expressions of mathematical entities and mechanical computation of algorithms, independently of their application, will be incorporated as an important part of the 'abstract' context constituting cognitive understandings of mathematics. Furthermore, it is assumed that schema theories that model cognitive structure (see below) can be used to investigate the 'harmful tangle' (Sfard, 1991) of 'fragmented' (Ball, 1990) and 'sparsely connected' (Simon, 1993) knowledge that preservice teachers exhibit within these abstract contexts.

Whatever the 'true' ontological nature of mathematical entities, our ways of talking, writing, and thinking of them requires that cognitive structures be developed in which they are granted at least nominal existence. As Sfard indicates: "Seeing a mathematical entity as an object means being capable of referring to it *as if* it was a real thing" (1991, p. 4, my emphasis). Here, it will be assumed that cognitive structure can be discursively investigated to determine if the abstract conceptual objects expressed and intended meaningfully refer to the ideal entities logically defined and implied by mathematical formalisms. In accord with constructivism, the development of understanding such conceptual referents entails that learners actively construct objects or schemas pertaining to them.

The theoretical framework assumed in this study is more fully explicated in Zazkis & Campbell (1994). Briefly, according to this interpretation, derived by Dubinsky (1991) from Piaget: *interiorizing* sensorimotor *activities* into *processes* that can be enacted in imagination which may then be *encapsulated* into atemporal cognitive *objects*, along with the construction of connections that relate disparate actions, processes and objects, lead to the *thematization* of more general cognitive structures called *schemas*. Novel experiences, if not readily *assimilated* by existing constructs, *dis-equilibrate* the subject which in turn may require and motivate alteration and further development of cognitive structure in order to *accommodate* some aspects of those experiences, thereby restoring *equilibration*.

Methodology

Individual clinical interviews were conducted with 21 pre-service elementary teachers; all volunteers from the group of students involved in a professional development course, "Foundations of mathematics for teachers." The interview protocol allowed for the flexibility to probe and clarify participant understandings of the concepts involved. The instrument was designed to present familiar concepts pertaining to division with remainder in novel and unfamiliar contexts. The instrument served to guide the investigation of epistemological obstacles and conceptual lacunas in our participants' existing content knowledge affecting their ability to assimilate and accommodate these problems. Calculators were available for the use of the participants. The specific question sets from these interviews, considered in this report and pertaining to whole number division with remainder, are:

Question set 1: If you divide 21 by 2, what would the quotient be? What would be the remainder?

Question set 2: Consider $M=3^3x5^2x7$. If you divide M by 15 what would the remainder be? What would be the quotient?

Question set 3: Suppose you're asked to perform division with remainder on 10561/24. Will your calculator help you? How?

Question set 4: Consider the number 6x147+1, which we will refer to as A. (a) If you divide A by 6, what would be the remainder? What would be the quotient? (b) If you divide A by 2, what would be the remainder? What would be the quotient?

The interviews were transcribed and categorized in terms of different questions, their difficulty, and identifiable cognitive tendencies of various degrees of sophistication exhibited by the participants. The analysis focused on relating division with remainder to the concepts of divisibility, division and multiplication. Only a brief synopsis of the results from this analysis can be reported in the space allocated here.

Qualitative data analysis and interpretation

Of the 21 participants considered in this study, 16 were asked to provide the quotient and remainder of 21 divided by 2. Most participants (12 of the 16) readily provided answers for the quotient as 10 and 1 for the remainder. However, responses from a couple of our participants, James and Louise, cogently illustrate two facts about division with whole numbers that can readily be taken for granted: first, that whole number division requires that the remainder and quotient are whole numbers; secondly, that the remainder be smaller than the divisor. Both of these cases will be considered in turn.

James' response to the remainder of 21 divided by 2 was .5. An analysis of the transcribed data with the worksheet he used through the course of the interview confirmed that he performed long division to obtain a rational quotient of 10.5 and identified .5, the decimal expansion of the fractional component of the rational quotient, as the remainder. When the interviewer asked James how he would conceive the remainder as a whole number he responded as follows:

James: We're dealing with 2 parts, so we'll add them together and get 1, remainder 1.

Interviewer: Okay. When you say that we're dealing with 2 parts, would you explain that a bit?

James: We're dividing the 20 into 2 parts. [...] So you're taking the 21 and, [I] don't know, it has a picture, you're taking it and splitting it in half and finding out what fits on either side.

Interviewer: Oh, so you've got 10 1/2 in here. . .

James: So you've got 10 parts here and 10 parts here, but you've still got that one part so half of it is over here, and half of it is over there. [...] So you've got 2 half parts, so you add them together, and you get 1.0. [...] And you get remainder 1. Look at it either way.

This excerpt reveals that James considered both division and the arithmetic units involved partitively. Despite the fact that the problem situations were presented in the context of elementary number theory and whole numbers, at least two thirds of the 21 participants in this study, on different occasions, identified a rational number as either remainder or quotient. Most of these cases led to considerable confusions that prevented participants from achieving task solutions.

With respect to the constraint that the remainder be smaller than the divisor, Louise, stated that the remainder was 3 with a quotient of 9, noting that '2x9 is 18, plus 3 is 21'. When the interviewer intervened to point out that the remainder is meant to be taken as less than the divisor she responded:

Louise: Right, I'm wrong. 10, with remainder 1 (laugh). I did that same mistake on the exam. Common error for me.

Interviewer: Why do you think you have a tendency to look at it that way?

Louise: Yeah, I went with the multiple closest, but the multiple closest would have been 10. I thought 9 was the closest to it, because of 18, I never even thought of 20.

Louise's quick reaction to the intervention and subsequent explanation indicates that her initial response pertained more to her familiarity with, or recall of, specific number facts rather than a gross misconception regarding the constraint upon the remainder. What is particularly evident in this case is that Louise did not perform long division in order to find the quotient and remainder. In looking for the 'closest' multiple, however, she revealed at least an implicit awareness of quotitive division. At least 10 of the 21 participants in this study, on different occasions and contexts, were readily disposed towards quotitive interpretations of division in problem contexts pertaining to whole numbers. The majority of these cases led readily to problem solutions.

Nineteen of the 21 participants in this study were asked what they thought the remainder would be if $M=3^3x5^2x7$ was divided by 15. Of these 19 participants, 8 eventually expressed a need to calculate in order to determine the remainder, while only 4, appealing to various divisibility criteria, were able to respond definitively that the remainder would be 0 (Zazkis & Campbell, in pressb). Of particular interest here is that at least 5 of the 19 participants asked this question (more than 25%) were unsuccessful in their attempts to assimilate this novel experience. They confused the remainder with the quotient in terms of 'what you are left with'. For example, one of our participants, Leigh, identified the remainder with 'the result', and then the quotient with 'the number of times' the divisor could be 'taken out' of the dividend. She confided: "I don't understand, I thought I understood the meaning of quotient, [...] when [the question is] set up like that I don't know how to take that meaning and apply it to that situation". A majority of the participants in this study experienced analogous difficulties assimilating and accommodating the meanings of quotient and remainder in the less familiar and potentially dis-equilibrating 'situational' contexts of question sets 3 and 4.

At a user level, division with calculators ostensibly involve the domain of rational numbers. If calculators are to be useful to enhance conceptual understandings of division, it seems crucial that they be effectively used for whole number division. There are two main approaches to determining a whole number remainder, R, from a rational quotient, q. The *integral* approach involves subtracting the product of the integral component of the rational quotient, [q], with the divisor, D, from the dividend, A: A-[q]D=R. The *fractional* approach involves multiplying the fractional component of the rational quotient, {q}, with the divisor: {q}D=R. On both intuitive and procedural grounds, it might seem that the latter approach would be easier to grasp or at least be easier to calculate than the former. However, the results of this study indicate that such assumptions could not be more mistaken.

Seventeen of the participants in this study were asked how their calculators could help them to determine the remainder in dividing 10561 by 24. Seven participants avoided dis-equilibration by indicating that a calculator would be of no help to them whatsoever. Four of these 7 participants were aware that in most cases division with a calculator would result in a value to the right of the decimal point that would be 'completely different' from a nonzero whole number remainder and could see no relationship between them. Some participants indicated that there was 'a way to do it' but they just couldn't remember or didn't know what it was. In fact, out of the 10 participants that fixated their attention upon the fractional component of the rational quotient, not one was able to successfully use the calculator to determine the exact whole number remainder. The difficulties and limitations of this approach gave way in almost all cases to our participants either assimilating the problem by resorting to long division in order to determine the remainder or giving up altogether.

In contrast, the experience of 7 of the 8 participants who used the integral method suggests that it is, conceptually and procedurally, a much more accessible approach to whole number division using a calculator. This appears to be due, in large part, to the fact that it evokes an intuitive quotitive interpretation that resists the conflation of numerical domains. The following excerpts from the interviews with Andrea and Kylie illustrate this approach:

Andrea: "10,561 divided by 24 equals 440.04167... 24 goes in here 440x24, so you go 10,560, with remainder 1."

Kylie: "Um, well the way, actually when I use a calculator to do um, to try things like this with larger numbers, um, you have to use it in a different way for it to help you, like you have to find out how many times; what I do with the calculator is, I find out how many times 24 goes evenly into 10,561 and then I multiply that number by 24, I see what the number, um, that comes up with it, it'll be slightly less ... either it will be even, it'll be the same as this, or slightly less, and you subtract um, you subtract the number [product of quotient and divisor] it [the calculator] comes up with from this [the dividend] to get the remainder."

It is particularly evident from Kylie's remarks that the integral method of finding a whole number remainder using a calculator evokes a quotitive interpretation of division that is clearly based upon the division algorithm: $A=QD+R$ where $0 \leq R < D$. Another important thing to note about this method of using the calculator for whole number division is that once the whole number quotient is obtained from the rational quotient, subsequent calculations exclusively involve whole numbers — not only avoiding problems pertaining to conflated numerical domains, but also interpretations of the fractional component as the whole number remainder (which is only valid when $R=0$), truncated decimal expansions and round-off errors observed in this study by those using a fractional approach.

The results of this study do not suggest that division with remainder will always be evident to learners when tacitly using the division algorithm. With question set 4, the dividend was explicitly expressed in the form of the 'right-hand side' of the division algorithm, and participants were requested to identify quotients and remainders. Of the 19 participants who were asked questions from this set, only 4 did not calculate the dividend for any of the questions they were asked. Of those 4, only 2 participants correctly identified the remainders and quotients for all the questions asked of them from this question set. The other 15 participants, at one point or another, resorted to calculating the dividend and dividing. Of those 15, 9 participants evaluated the dividend and relied upon long division in addressing this question set. And, finally, 2 of those 9 participants initially attempted alternative procedural activities that both led to the rational quotient; both, in fact, due to difficulties interpreting the fractional component. Ironically, 15 of 19 of our participants, at some point in question set 4, resorted to a complex iterative application of the division algorithm — long division.

Conclusion

The division algorithm establishes an important structural relation and defines the operation of division amongst the whole numbers, the elements of which are ostensibly referred to by the terms: dividend, divisor, quotient and remainder. Within this context, the relation between these terms can be clearly and unambiguously defined. The phenomenological origins of the division algorithm appears to reside in an intuitive understanding of indivisible units. Whereas the motivation for the division algorithm can be seen as a natural consequence of such a constraint. These data strongly suggest an implicit relation between our participants' fixations upon fractional units and partitive division and integral units and quotitive division. The distinction between fractional and integral units, I believe, is prior to the partitive and quotitive models of division. Difficulties reconciling formal and intuitive models of division may stem from compromised thematization of division schemas in part due to conflation of the basic conceptual objects distinguishing whole and non-negative rational numbers.

Be this as it may, it is a common didactic practice to consider whole numbers, for all practical intents and purposes, as a subset of rational numbers (e.g., Freudenthal, 1983, p. 103). This practice, which promotes and advocates dual emphases upon the numerical aspects of count and measure, the equivalence and equivocality of 'how many' with 'how much', implicitly carries with it the implication that any particular whole number is also a rational number and therefore can, both in principle and in practice, be divided. Conflating these aspects of number trivializes the historically and conceptually crucial distinction between divisible and indivisible units in pre-Euclidean Greek mathematical philosophy. More importantly, such conflation appears to raise the potential for conceptual confusions and semantic ambiguities with respect to the procedures and numerical reference of terms pertaining to division in different domains.

Even if the numerical domains are not conflated and properly referenced with respect to the operation of division, there is another layer of problems that arise with respect to less familiar expressional forms of the numbers involved. As we have seen, whole numbers can be expressed not only in their familiar decimal representation but also in the potentially dis-equilibrating forms of prime decomposition and equational expressions such as the 'right-hand side' of the division algorithm: QD+R; e.g., $45=3^{2}x5=11x4+1$. Furthermore, rational numbers can be expressed in decimal representation with expansions 'to the right of the decimal point' as well as in various fractional forms; e.g., 11.25=11

1/4=11 + 2/10 + 5/100. The data from this study identify significant obstacles and lacunas in some learners' understanding of the meaning and reference of terms pertaining to division involving these different expressional forms. The nuances of the data set and the theoretical analysis of the schemas involved are too complex to explicate any more details within the space provided.

Most learners of mathematics are tacitly familiar with the division algorithm through 'performing' and 'checking' the results of long division. It is indeed tempting to consider the division algorithm itself as the 'inverse' of whole number division in the same sense that one would consider multiplication as the inverse of division in the domain of rational numbers. However, an inverse relation of whole number division with multiplication obtains only in the restricted sense of divisibility (Zazkis & Campbell, in pressa). Even with an understanding of divisibility, understanding the nature of whole number division and it's relation to multiplication is not readily evident. As one of our participants observed: "you never have a remainder in multiplying. [...] an inverse of something is just the opposite. Once you have a remainder it makes it totally different from multiplication." The question that begs to be addressed by teachers, learners and researchers alike is WHY?

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REFERENT TRANSFORMING OPERATIONS: TEACHERS' SOLUTIONS

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Referent transforming operations are difficult for pupils and some indication already exists that this may also be so for teachers. Some pupils succeed in these problems by using informal strategies that allow them to keep the situation in mind. We investigated the responses of 40 Brazilian teachers to two problems involving referent transforming operations and analysed their approaches when the solution was successful. Modest rates of success were observed; upper-school teachers performed better than primary school teachers. The majority of the successful solutions was obtained through formal strategies. These results raise the questions of whether the teachers' knowledge of street mathematics might emerge in other situations and how to integrate experiential and formal knowledge of mathematics in teacher education programs.

The difficulties of referent transforming operations have been widely discussed in the literature about children's conceptual development. Prospective teachers also have difficulties with such transformations of referents (Simon, 1993) when the transformations are not of the well practised sort such as distance divided by time equals speed. When we consider that one of the important goals of numeracy development in school is to make mathematical approaches to problem solving available to students, teachers' difficulties with referent transforming operations become a clear stumbling block for the accomplishment of this goal. How can we solve a problem using mathematics if we fail to understand the meaning of the answer?

Researchers in different countries have already documented primary school teachers' difficulties with division and multiplication problems (e.g. Simon, 1993; Tirosh & Graeber, 1989) and with referent transforming operations, where intensive quantities were involved (Thompson & Thompson, 1994). It is thus clear that teacher education needs to effectively tackle the question of promoting the understanding of more complex models in the conceptual field of multiplicative structures.

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In order to design strategies for teaching teachers, we investigated the nature of their difficulties and the solutions found by the successful teachers in two problems, one involving multiplication and the other division. Because students have been shown to keep the meaning of problem situations in mind by resorting to non-algorithmic solutions (such as building-up strategies in ratio problems), we wanted to know whether these types of solution would also be predominant among the successful teachers or whether their (assumedly) better mathematical knowledge would allow them to achieve meaningful solutions through more formal routes. The worst scenario would be one where the teachers did not use non-algorithmic solutions (perhaps because these are somehow excluded from their social representation of adequate mathematical knowledge) but have not developed enough formal mathematical knowledge to implement formal solutions correctly. Thus we decided to investigate also the impact of training routes on their relative ease in handling referent transforming compositions.

Method

Subjects: We interviewed 20 Brazilian primary school and 20 upper school teachers in São Paulo. The two groups differ both in terms of teacher education and level of instruction in mathematics. Primary school teachers are educated at the secondary level in a three-year program. They teach all the primary school subjects and consequently cannot develop their studies of mathematics in depth. Upper-school teachers (for grades five and beyond) are specialized and follow a four year university program with considerable amounts of study of their own subject plus teacher education courses.

Design: The teachers were asked to participate in a study to investigate their ideas about some mathematical concepts and how to explain them to pupils. They answered a series of questions, clearly identified as related either to multiplication or division. This reports only analyses two of their responses.

We chose two problems expected not be common in the classroom (see Table 1); one proposed a new situation (problem 1) and the other asked a different question from what would be usual for fractions problems (problem 2). Both problems dealt with extensive quantities to avoid confounding the difficulty of referent transformation with that of intensive quantities. The problems were adapted from Simon (1993; Simon and Blume, 1994) to allow for comparisons.

Table 1

Problem 1. John and Paul worked together to measure the sides of a rectangular surface. John measured the width and Paul measured the length. Each one used a rod and their rods were of different sizes. John said: "The width is 5 of my rod". Paul said: "The length is 4 of mine". What could they conclude about the area of the surface? Why?

Problem 2. Serge has 35 cups of flour to make muffins with. For each muffin he needs 3/8 of a cup. If he uses all the flour he has to make all the muffins he can, according to the recipe, how much flour will be left?

Procedure: The teachers responded to the questionnaire in writing. The interviewers were supportive when the teachers expressed difficulty and concern over their own mathematical knowledge but did not offer further clues.

Results

Quantitative analysis: First, the overall performance of the two groups was compared. We do not make any assumptions about the representativeness of these samples, which are small and were not drawn in a systematic manner. The quantitative results can only inform about the sample itself and no generalizations are possible. Table 2 shows the frequencies of teachers who did not attempt the problem, those who attempted but failed to reach a correct answer, and those who succeeded.

A correct response in problem 1 indicated the number and nature of the units of measurement of area; in problem 2, 1/8 of a cup was left.

Table 2

Frequency of responses for each group of teachers

	Problem 1			Problem 2		
	Blank	Wrong	Right	Blank	Wrong	Right
Primary school	10	8	2	8	12	0
Upper school	0	11	9	1	12	7

The quantitative analysis indicates that: a) the problems were not trivial for the teachers and rates of success are modest (in the case of upper-school teachers, the rate of success in problem 2 is comparable to that observed by Simon, 1993, in his sample of U.S. primary school teachers); b) primary school teachers attempted the problems much less often, even when no computational difficulty was present (problem 1); c) the difference in performance across the two groups in these items was significant (a t-test for independent samples indicated that the mean correct responses for both problems differed significantly at the .004 level).

Qualitative analysis: The purpose of this analysis was to investigate the nature of the successful strategies. For each problem, we devised a descriptive classification of the process used in calculating the answer.

Responses to problem 1 were analysed into the four following categories.

1. **Multiplying the numbers and the units, indicating that a new unit is formed by the product of the original units.** Two different approaches were identified: a formal one, where the units and their product were represented by letters; and an integrated approach, where a formal representation was related to an experiential one, with explicit mention of a rectangular unit of area. It is noteworthy

that no experiential solution without a formal approach was observed. No responses from the primary school teachers and 9 of those from the upper school teachers fell into this category. An example of the formal approach is:

$5j \cdot 4p = 20\,jp$ j (measure of one rod) p (measure of other rod)

An example of an integrated approach is illustrated in this answers, where two parallel solutions were offered:

20 times the product of the two units of measure

1 : For each rod L there are 4 in the length and for each rod C there are 5 in the width, that means, there will be 20 rectangles equal to L and C inside the whole figure

2: $L \to 5 \cdot uj$

$C \to 4 \cdot up$

$L \cdot C = A,$

$5 \cdot uj \cdot 4 \cdot up$

$20\,uj.up = 20 \cdot$ *prod. unit of measure*

2. **Multiplying the numbers but making no reference to the units, as if the issue were not problematic**. Two of the responses from each group of teachers fell into this category.

3. **Denying the possibility of calculating the area**. Some of the teachers denied that it was possible to know anything about the area of the surface because, they argued, in order to measure, it is necessary to have a single unit. This explanation clearly fails to take into account that a new unit would be formed by the product of the two linear measures and would be constant for the whole surface. Three primary school and five upper-school teachers gave this sort of response.

4. **Responses which lacked internal consistency**. Some responses seemed to reflect the consideration of the emergence of a new unit but were incomplete or inconsistent with the information in the problem. Four primary school and five upper-

school teachers showed this type of response. One example:

> *20 square rods; it is a rectangular region (our underline; notice that the teacher indicated that a new unit is formed but calls it "square", apparently without a full realization of both senses of "square" in this expression, that the unit is squared because x is multiplied by x and that the resulting area unit is a square).*

Responses to problem 2 showed a greater variety of approaches, which we attempted to capture in six descriptive categories.

1. **Uses a formal procedure and maintains the transformation of the referent in mind.** This route involves dividing 35 cups by 3/8, identifying how many muffins can be made and what portion of a muffin is left; this calculation is followed by a new transformation of referent, from fraction of muffins into fraction of a cup. The only two upper-school teachers who used the formal solution successfully wrote down the referent at the end of each computation, as exemplified in this transcript:

35 cups 1 muffin 3/8 cup

no of muffins = $\frac{35}{3/8}$ = *35 x 8/3 = 93 1/3 muffins (calculations here)*

93 muffins 1/3 muffin left

1/3 muffins 1/3 x 3/8 cup = 1/8 cup left

2. **Uses a formal procedure but fails to observe the transformation of the referent.** This strategy (used by 3 primary-school and 7 upper-school teachers) involved the use of the same initial step of dividing 35 by 3/8 but the teachers took the remainder 1/3 to refer to cups of flour.

3. **Uses a measurement solution and thereby avoids the transformation of referents.** This solution (used by two upper-school teachers) involves calculating how many eighths in 35 wholes, followed by a division by 3/8. One example:

$$35 = \frac{35 \,.\, 8}{8} = \frac{280}{8} = \frac{279}{8} + \frac{1}{8} = \frac{3}{8} \,.\, 93 + \frac{1}{8}$$

4. **Uses a scalar solution and keeps the referent in mind.** This solution (used by two upper-school teachers) involved a building-up strategy where the correspondences between cups of flour and muffins were preserved throughout; shortcuts were used rather than a laborious composition muffin by muffin or cup by cup. For example:

1 cup → 2 muffins and 2/8 left
2nd cup + 1/8 → 2 muffins + 1 muffins + 1/8
3rd cup → 2 muffins + 2/8   1 muffin
3 cups → 8 muffins
33 cups → 11 x 8 = 88 muffins
2 cups → 5 muffins and 1/8 cup left
93

5. **Calculates 3/8 of 35 and indicates either this or 5/8 as the answer.** This appeared to be the solution of a traditional fractions problem - What is 3/8 of 35? or: What is left from 35 if you take 3/8 away? - rather than an answer to the problem at hand. In either case, unreasonable responses are obtained: the answers are approximately 13 and 22 cups, which represent amounts considerably larger than 3/8 of a cup, the upper limit for answers to this problem. One primary school and three upper-school teachers responded in this way.

6. **A mixture of computations that we could not interpret in any model.** A total of eight primary school and four upper-school teachers presented this type of response.

Discussion and conclusions

The main purposes of this study were to investigate whether referent transforming operations pose difficulties for teachers and how they surpass these difficulties. In problem 1, few teachers established a clear connection between the product of two linear measures and a unit of area. In problem 2, many failed to realize the inadequacy of solutions where the answer was larger than 3/8. These results suggest that it is possible that non-algorithmic solutions are blocked by school instruction in mathematics even for teachers, who might be expected to succeed if they had tried out such solutions. Had the teachers - many of whom were females - been concerned with problem 2 in their kitchens, would they have behaved differently?

Although this study answers some questions, it raises many questions for further research in teacher education. First, it would be of interest to know whether work with teachers would show effects of the social context of testing similar to those observed by Nunes, Schliemann, and Carraher (1993) with Brazilian youngsters. A positive answer would represent a challenge to teacher education: Can teachers learn more mathematics in their teacher education without losing touch with their knowledge of street mathematics? Can most be taught in such a way that they will be able to connect their experiential and their formal solutions? And, if so, will they then devise techniques for instructing their students which will promote such connections? Second, the variety of approaches observed in problem 2 raises another question: Are teachers in a position to assess the different approaches if their pupils were to use them? Can they recognize the different ways of reasoning underlying the different methods? And what role might the discussion of these different methods play in teacher education?

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RECONSTRUCTING TEACHERS' THINKING IN TEACHING PROBLEM SOLVING[1]

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This paper reports on an open-ended, experiential problem solving inservice program and its effects on teachers' thinking and teaching of mathematical problem solving. The program was framed in a constructivist perspective of learning. The participants consisted of 6 elementary teachers who disliked/feared problem solving. Data collected by interviews and classroom observations were used to identify patterns in the teachers' behaviours and attitudes to determine the effects of the program. The results indicated positive changes in the teachers' confidence in their ability to solve problems and in their teaching approaches. They also suggested that allowing teachers to work from personal experiences to self-constructed theories of practice is a meaningful way of helping teachers to transform their teaching to reflect the philosophy of current reform movements in math education.

Changes that are now being advocated in mathematics education (NCTM, 1989, 1991) require teachers to abandon their traditional transmission approach to teaching mathematics for more innovative approaches. Given the importance for mathematics teachers to change to facilitate the successful implementation of such reform recommendations, research studies involving the mathematics teacher's knowledge, beliefs, practices and learning have begun to gain prominence (Cobb, Wood & Yackel, 1990; Cooney, 1985; Hoyles, 1992; Ponte, 1994; Simon & Schifter,1991; Thompson, 1984; Wood, Cobb & Yackel, 1991). One of the underlying themes of these studies is to understand how to influence change, where necessary, in the teaching of mathematics. The issue of change raises the issue of the nature of inservice teacher development programs to facilitate it. Cobb, Wood and Yackel, for eg., in their work have focused on teachers' change in the context of classrooms as learning environments for teachers and teachers' learning is tied to the classroom interactions.

The current trend to accept a constructivist perspective of learning (von Glasersfeld, 1991) in mathematics education is also flowing over into teachers' learning as reflected in the Cobb et al and Simon & Schifter studies. The constructivist perspective recognizes that knowledge cannot be acquired passively and can only take place when existing cognitive structures meet with perturbations. von Glasersfeld (1989) emphasized that the most frequent source of perturbations is the interaction with others. Placed in this context, mathematics teachers' learning can be viewed as both an individual and an interactive activity during which the

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teachers construct their own meaningful knowledge by engaging in and reflecting on experiences with mathematics and negotiating personal meaning of these experiences with peers. Thus learning can be characterized as a process of self-reflection and mutual adaptation by which the teachers organize their beliefs to give meaning to their personal experiences in the process of interacting with others. The study presented in this paper is framed in this perspective of constructivism and teachers' learning.

The study focused on mathematical problem solving, an area that has been under-represented in research in terms of the role of the teacher in teaching problem solving (Silver, 1985) and continues to be so. Given the significantly increased emphasis on problem solving in the current reform recommendations, providing help for teachers to effectively deal with this, becomes critical. Such help is probably more important for elementary teachers who are more likely to have poor backgrounds in mathematics and to conceptualize problem solving through the eyes of a "poor" problem solver. An approach based on constructivism becomes relevant in working with these teachers to provide them with meaningful experiences to allow them to reconstruct their understanding of problem solving. Inservice programs for teachers have traditionally been prescriptive, an approach that is now viewed as problematic in facilitating meaningful outcomes for the teacher. In shifting to a constructivist perspective, this study was based on an inservice program that did not provide a "recipe" for doing or teaching problem solving. It was assumed that the teachers would construct their own processes by themselves. The outcome of the study determined the extent to which this occurred and its nature.

RESEARCH PROCESS: Six elementary teachers (grades 3 to 6) were selected from those who volunteered to be participants of the study. The criteria for selection were that they not be mathematics majors or mathematics specialists and that, as students, they had a fear and dislike of mathematics, in general, and problem solving, in particular, and continued to feel this way. The teachers were all exposed to problem solving based on the approaches prescribed by the textbooks they used. These approaches generally involved the "routinizing" of non-routine problems by demonstrating a strategy, then presenting several problems that could be solved using that strategy. This resulted in the teachers presenting problem solving in an algorithmic way that focused on getting the "right" answer. The study, conducted as a descriptive qualitative study, was carried out in two parts. The first part consisted of the problem solving inservice

(PSI) program and the second, of data collection to determine the effects of this program on the teachers' thinking and teaching of problem solving.

(a) The PSI Program: In this program, problem solving is being used as the process of solving non-routine or open-ended mathematical problems. The program involved engaging the teachers directly and indirectly in problem solving activities for 20 hours "in-class" and at least 4 hours/week on "take-home" assignments over a 4-week period during their summer break. The researcher (a mathematics education professor) was the facilitator of the "in-class" activities. The key ideas that provided the framework of the program were:

i. Teachers should critically examine, through their personal experiences, the nature of problems and the process of solving problems as a basis for understanding how to teach problem solving.

ii. Teachers should be placed in a learner's role to experience genuine problem solving in a setting which fosters individual and social construction of problem solving concepts.

iii. Teachers should reflect on both the affective and cognitive aspects of their problem solving behaviours to understand problem solving in a more realistic and humanistic way.

The key activities of the program included the following: The teachers,

- shared how they taught problem solving.
- shared and resonated in stories of personal experiences that reflected their feelings and beliefs about problem solving.
- shared and reflected on how they solved problems in the real world.
- solved non-routine and open-ended problems individually and in groups of 2 and 3.
- working in pairs, observed each other solve problems while thinking aloud.
- working in groups of 3, took turns being the teacher and students while solving problems.
- taped themselves as they individually solved a problem thinking aloud and including all emotional aspects of the experience.
- solved problems by writing detailed narrative journals of their processes, including all emotional aspects of the experience.
- in all of these problem solving situations, reflected on several aspects of their experiences, including their problem solving processes, their feelings, their thinking, the relationship between their processes and their real life problem solving processes, what they now knew

that they did not know before, and how should problem solving be taught. These reflections were done orally, as a group, and were audio taped.

In general, for the purpose of the study, all oral aspects of all of the activities were audiotaped and all tapes were transcribed. Copies of all written work were also obtained for data. At the end of the program, the teachers were given readings on how to assess problem solving.

(b) Other Data Collection: The second part of the study consisted of observing the teachers in their classrooms teaching problem solving to collect information to determine the effect of the program on their teaching. They were observed a total of four hours each, one prior to and 3 over a period of 1 1/2 years after participating in the PSI program. Field notes were made during the observations and the lessons were audio-taped. Each observation was followed by an in depth, open-ended interview on their teaching, beliefs and attitudes towards mathematical problem solving. All interviews were tape recorded and transcribed.

Data analysis involved a thorough examination of the data to identify patterns in the teachers' behaviours and attitudes to determine the nature of the effects of the PSI program.

RESULTS: The results indicated positive changes in the teachers' attitudes and teaching of problem solving. Positive changes are being interpreted as a shift towards recommendations of the NCTM standards. What follows is a sample of some of these changes.

Effects on attitudes: The most significant outcome for the teachers was their increased confidence in their ability to solve both routine and non-routine problems within the context of their teaching and the active range of their mathematical background. The teachers attributed this to becoming aware that many beliefs they had of themselves as problem solvers and of the process of mathematical problem solving were misconceptions and created obstacles in their learning. Some of these beliefs were: "You **must** follow an algorithm to solve a mathematics problem." "You shouldn't abandon a method while solving a problem. If it wasn't working, it was because something was wrong with you and not the method." "Getting stuck and frustrated was proof of how dumb you were in solving mathematics problems." Interacting with peers and seeing each other "struggle" with similar things in solving the problems made them understand their personal struggles with problem solving in the past in a more objective way. Comparison to their experiences with solving real life problems also contributed to their understanding of situations like "barriers" and "frustrations" as integral features of problem solving and not situations they "create" because something was wrong with them. Successfully solving problems they thought they could not do, with little or no help from anyone, but depending on their own thinking, conflicted what they were told by their math teachers and verified what they always felt "deep down inside" -- that they could do better in solving mathematics problems. In general, their awareness seemed to free them from the traps of their past experiences and gave them permission to do things they thought were not allowed or valid when solving mathematics problems. It also shifted their view of problem solving from a prescribed algorithmic process to an open-ended process in which the problem solver had to be in control in terms of interpreting the problem and deciding on how to overcome the inherently necessary barriers to a solution. This view was reflected in their teaching.

Effects on teaching: The program also had a significant positive effect on their teaching approaches. There were several aspects of their experiences with the program that they integrated into their teaching. However, although there were many similarities, each teacher lived out the experience in her classroom in her own unique way. Common aspects of their behaviours included, a shift to: engaging students in problem solving more often; being less dependent on a textbook; using more cooperative learning groups; having students share solutions and meanings; emphasizing process over final answer; listening more to students, focusing on the students' thinking behind "right" and "wrong" answers; using non-routine problems; exploring alternative solutions; asking non-leading questions and being more sensitive to the nature of intervention in the students' processes, i.e., when and how they provided help. These were all behaviours they conceptualized as a group based on their collective experiences with the program. In fact, they had constructed some theories to guide their teaching. For example, they identified 3 situations that warranted active intervention by the teacher: stuck, off-track and lost. The nature of the intervention was different for each of these state. The overriding principle was that intervention could not be to tell students how to get the answer but to stimulate their thinking to get over barriers and to make sense of what they were doing. Most of the interventions observed during their teaching were at the stuck state and often consisted of the teachers asking open-ended questions.

The differences in the teachers' behaviours are much more difficult to summarize because they are related to how the teaching process unfolded for each teacher and should be presented in this context. Some of the teachers, for example, emphasized the meaning of "barriers" and "stuck" and framed their teaching around these concepts. Others focused on "does it make sense?" Some developed a structure to help students to attack the problem while others were more open ended in their approaches. However, in general, they altered their teaching in ways that strengthened where they thought there were weaknesses based on the knowledge they had now acquired about problem solving. Given the constraints on space, it is not possible to describe each teacher's situation. Thus two cases of teaching structures and some excerpts of their post-program experiences are summarized here.

Mary (grade 3), prior to the PSI program, did not engage her students in genuine problem solving. She focused on teaching algorithms based on key words and assigned problems that could easily be solved using those steps. She would read the questions to emphasize the key words for the students. Students worked individually, but were expected and guided to solve the problem in the same way, following the algorithms. About a year after the PSI program, her teaching looked as follows: She wrote the problem on a flip chart. Students gathered around the chart sitting on floor. She asked them to read the problem to themselves. She asked one student to read it aloud. She asked them if they saw the difference between the information part and the question part of the problem. She asked a student to read the question part. She checked on whether the others agreed. She asked another student to read the information part. She asked if the information made sense, or, what did they think about the information. She gave them time to solve the problem working in small groups. She circulated and used non-leading questions to intervene when they needed help. Students regrouped to present and discuss their solutions. She first asked them to share something they liked or did not like about the problem. She then allowed them to present their solutions and to explain why their solutions made sense. She reminded them to listen to the other ideas to see if they made sense.

Kate (grade 4) described her approach prior to the PSI program as follows:

> I tended to just give it (the problem) to them and then leave them with it and then at the end the ones that were successful would present their answers and the others would look at it.

She usually started with a structured format of the Polya's 4-stage model and a list of strategies prescribed in the text she used. She explained:

> Last year I would make them (the students) write what strategy they were going to use and then do it. And they would often have the wrong strategy. They would say, "I'm going to do so" and then they do something different. I don't think it was helping them.

The year following the PSI program, she ignored the structured format of Polya's model and delayed going through a list of strategies until later in the school year. Instead, she focused on getting the students to think. This she found to be a challenge. She explained:

> This class is really resistant to thinking. I now make them think, but my two top kids, for example, hate me for that, because now they are not the ones coming up with the answers, because they are use to looking at things obvious to them and so they feel frustrated. But some of the weaker kids are doing much better.

In her teaching, first she asked the students to read the problem and to write their meanings or interpretations of the problem. She then asked a few volunteers to read theirs and, if there were no questions about surface features of the problem, instructed the class to solve the problem. She circulated and intervened with non-leading questions only if the students indicated to her that they were stuck. After solving the problem, the students were required to present their solutions, regardless of whether they were "right" or "wrong", and defend them to the class. The goal was for the students to decide on what made sense and why.

The grade 6 teacher summarized some of her changes:

> This year I have really concentrated on not jumping in for them and saying, "Yes, that is right! That is right!" because I used to feel so happy that they were getting close to the answer. Now, instead, I will question them and say, "Well, what do you think?" "What about the looking back stage?" "What are you going to do now?" So I try not to wrap it up quite so nicely for them. We will eventually go over the solution or solutions. I think I have made them and myself more aware of other solutions that could be available. I am making them more aware of looking at things from different angles, from different perspectives, not only in their answering but in their figuring it out as well.

Another grade 6 teacher noted:

> Well I think more kids are feeling successful. Maybe it is because I feel more successful too and maybe I think it is more fun now too. I do think that it is more fun. And so that is bound to rub off on other kids too. And also the business of hearing how they did it. It is fun to see their ways and to see that they can be quite different but still very acceptable.

Similarly, the grade 5 teacher noted:

> Well it is interesting, I had a comment a couple of weeks ago from a girl who said, "When are we going to do problem solving?" as if it was a separate entity. And I said, "Well, we have been doing problem solving all along." But, so in a way I guess that it is more fun for them, that they don't realize what they are actually doing sometimes.

All of the teachers pointed out that their teaching after the PSI program was more challenging, but more interesting and rewarding particularly because they were learning a lot from the students. They also found that the way they were teaching other areas of mathematics was being influenced by their problem solving approaches.

CONCLUSION: The outcome of the study indicated that an open-ended, experiential problem solving inservice program, grounded in a constructivist philosophy, can make a significant contribution in helping teachers to reconstruct their thinking and teaching of mathematical problem solving and mathematics in general. Although the effectiveness of the program was probably helped by the fact that the participating teachers were highly motivated to revise their teaching and participated of their own free will, what seemed of significant importance, was allowing the teachers to work from personal experiences to self-constructed theories of practice. This seemed to have allowed them to construct meanings that reflected theories that made sense to them in the context of their teaching. Overall, this seemed to be a more meaningful way to the development of mathematics teachers to better help them to transform their thinking and teaching to reflect the philosophy of recent reform movements in math education.

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Bayesian implicative Analysis : How to identify the concepts necessary to acquisition of a competence ? One example of application to the conceptualization of fraction in teenagers.

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Does success to test A imply, in general, success to test B ? Such question about oriented dependencies often arise in Psychology of Education. Loevinger's index H can be used to measure descriptively such near-implications. A bayesian methodology is proposed to generalize conclusions reached after this descriptive stage. Results of the method are summarized by an implicative graph relating several tests. This Bayesian Implicative Analysis is used for analyzing an experiment about the conceptualization of fractions in teenagers.

For a psychologist specialized in mathematics education, it can be very useful to have at his disposal methods allowing him to identify the notions necessary for the acquisition of a competence in a given conceptual field. With such methods, one could evaluate the effects of a domain already studied and could thus contribute to the development of a new didactic. Before acquiring a concept, it is necessary to go through several stages that sometimes include epistemological obstacles (Vergnaud, 1981).

A difficulty appears during the acquisition of rational numbers : fractions expressing a non-inclusive relationship (Part/Part) are more difficult to master than fractions expressing an inductive relationship (Part/Whole) (Vergnaud, 1983). Then, two questions arise : 1) To what extent the Part/Whole fractions are necessary to master Part/Part fractions ? 2) What are precisely the obstacles faced during the conceptualization of Part/Part fractions ? The purpose of this article is to show how it is possible to answer these questions through an example of research about the conceptualization of fractions during secondary school with the Bayesian Implicative Analysis.

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From experimental data taken from a group of students, it is possible to quantify the oriented dependencies between the performances (success or failure) observed during two tests. An near-implication (or in short an implication $a \rightarrow b$) of a modality a towards a modality b appears when the students presenting a present generally b, whereas the opposite is not necessarily true. Therefore, it is important (1) to have an index that quantifies descriptively the magnitudes of near-implications, for a sample of subjects, (2) to have inductive methods that generalize these near-implications to the reference population from which the sample was extracted. The Bayesian methods allow to deal with this question in a general way (Bernard, 1991). They have been recently applied to the implicative analysis (Charron, 1996 ; Bernard et Charron, submited).

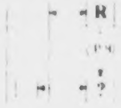
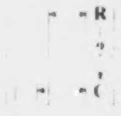
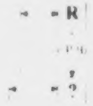
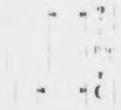
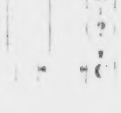
Method

18 problems concerning the use of fractions were given to 165 pupils : 55 fifth grade pupils (average age : 10 years and 11 months), 55 seventh grade pupils (average age : 12 years and 10 months) and 55 ninth grade pupils (average age : 15 years and 3 months). In each problem, a fraction (given or to be found) was relating a compared quantity to a quantity of reference. There were three different type of tasks : computation of the fraction (OF), computation of the compared quantity (QC), computation of the quantity of reference (QR) in two different cases (the fraction expresses a Part/Whole relationship, PW, or a Part/Part relationship, PP). For each of these six types of problems, three situations were presented (slices of cake, clients in a restaurant, trees in a forest). The crossing of the « task », « relationship » and « situation » factors led to the elaboration of 18 problems linguistically similar. Examples of problems are presented in table 1 :

Results

Each of the six types of the problems above was considered as a single test. Each test was coded in success/failure according to the following criteria : success (denoted 1) when at least two of the three situations were correctly answered, failure (denoted

Table I : *Statement examples for each of the 6 tests.*

	Computation of the Compared Quantity of the Part-Whole Relationship (QC/PW)
	"In a forest, there are 80 trees and [illegible]/5 of the trees are chestnut trees. Find the number of chestnut trees in the forest."
	Computation of the Reference Quantity of the Part-Whole Relationship (QR/PW)
	"At a restaurant, 30 clients have finished eating, that is 3/5 of all the clients in the restaurant. Find the number of clients in the restaurant."
	Computation of the Fraction of the Part-Whole Relationship (OF/PW)
	"A large cake is made up of 90 slices. 36 slices have been eaten. What fraction has been eaten? Reduce the fraction to the simplest form possible (a fraction that can not be reduced any further)."
	Computation of the Compared Quantity of the Part-Part Relationship (QC/PP)
	"At a restaurant, 70 people order meat and the others order fish. The number of people who order fish represent 2/5 of the people who order meat. Find the number of people who order fish."
	Computation of the Reference Quantity of the Part-Part Relationship (QR/PP)
	"In a forest of cedar and pine trees, there are 60 cedars. That is, 3/5 of the pine trees. Find the number of pine trees."
	Computation of the Fraction of a Part-Part Relationship (OF/PP)
	"A cake has 49 decorated slices. 14 slices are not decorated. What fraction represents the number of non-decorated slices, compared to the number of decorated slices? Reduce the fraction to the simplest form possible (a fraction that can not be reduced any further)."

0) otherwise. During a preliminary analysis, no effect of surface traits (cakes, clients, trees) was discovered. Thus, we focused on the study of the global implicative structure relating these six tests : QCPW, QRPW, OFPW, QCPP, QRPP, OFPP. The implication relations between these tests, taken two by two, have been quantified by the Loevinger's index, that ranges from -∞ to 1 (see Charron, 1996 and Bernard and Charron, submited, for more details). In the following, an observed Loevinger's index, denoted H, should be clearly distinguished from the corresponding « true » index, denoted η, that is the one in the population. There is no implication for a negative values ; for a value of 0 the two tests are independant ; the closer to 1 the H index is, the higher the implication is ; there is a strict implication for a value of 1. In practice, we define a reference value, called « negligible limit of implication », under which we consider that the index value is negligible. On the contrary, a « limit of notable implication » is also defined, above which the implication will be termed as notable. As far as our set of problems is concerned, these reference values were respectively taken to be .20 and .60. The following tables present observed relations of vary in magnitude (in terms of H) for the ninth grade students, respectively from QCPW to QRPP, from OFPP to QRPW, from QRPP to OFPW, and from QRPP to QRPW. In each case the first test is in lines, the second one in columns :

	1	0
1	14	31
0	3	7

H 0,002

(negligible implication)

	1	0
1	20	5
0	12	18

H 0,521

(intermediate implication)

	1	0
1	16	1
0	22	16

H 0,909

(notable implication)

	1	0
1	17	0
0	15	23

H 1

(strict implication)

The figure 1a presents the descriptive implicative graphs which reflect the near-implication relations between the six types of problems, for each school grade. Only the non negligible relations (that is the ones such as H > .20) are represented. The values of the H index appear besides each line. The line's thickness reflect the value of H. The arrows indicate the implications between the successes and the dotted lines indicate the implications from success to failure (negative exclusion) or from failure to success (positive exclusion).

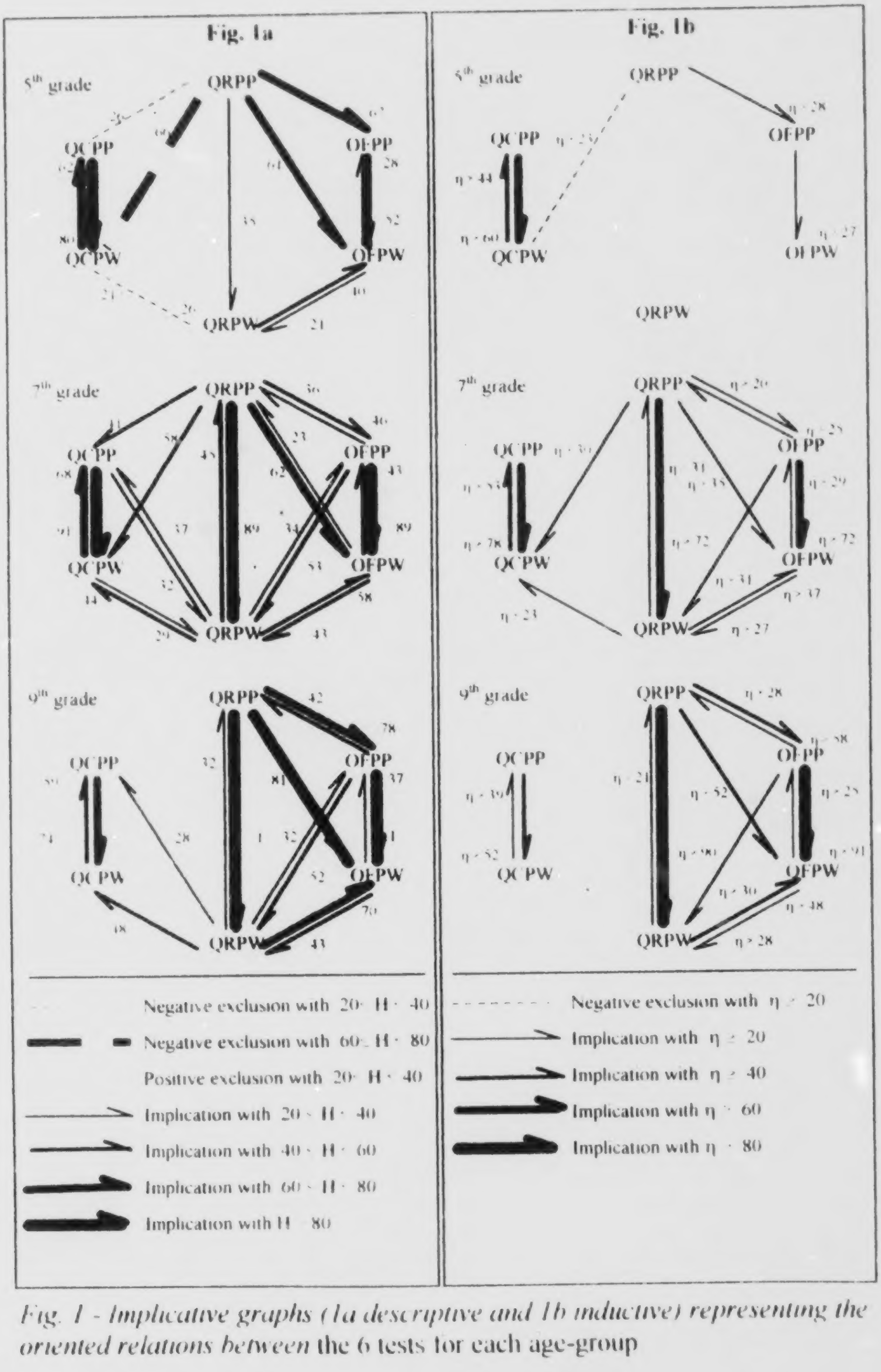


Fig. 1a
Fig. 1b
5th grade
7th grade
9th grade
QRPP
QCPP
OFPP
QCPW
OFPW
QRPW
Negative exclusion with 20< H < 40
Negative exclusion with 60< H < 80
Positive exclusion with 20< H < 40
Implication with 20 < H < 40
Implication with 40 < H < 60
Implication with 60 < H < 80
Implication with H > 80
Negative exclusion with η > 20
Implication with η > 20
Implication with η > 40
Implication with η > 60
Implication with η > 80

Fig. 1 - Implicative graphs (1a descriptive and 1b inductive) representing the oriented relations between the 6 tests for each age-group

The figure 1b sums up the cases for which the Bayesian inference relative to the index η allows to conclude inductively to a non-negligible implication. The values η_0 such that Prob($\eta > \eta_0$) and $\eta_0 > .20$ are written besides each line. This type of inductive statement means that there is a 90% probability that the index η is higher than η_0. The type of lines, solid or dotted, must be interpreted as in figure 1a. Thickness now indicates the value of η_0. The fact that some lines disappeared from 1a to 1b corresponds to the situations of ignorance about the associated η's. For these cases, it would be useful to have a higher number of students to reach a conclusion. In this respect, the descriptive graph is a guide that indicates the relations that could be demonstrated if more data were available.

To be concise, the commentary is focused on the inductive graph (1b). Nevertheless, the reader can refer to the descriptive graph, particularly in cases of ignorance. Figure 1b shows that the implication network is much richer for seventh graders and that a success to failure implication (negative exclusion) only appears for fifth graders. Such negative exclusion between QCPW and QRPP means that most of the pupils who succeed in resolving the first problem, failed on the second one. For seventh graders, this relation is replaced by an implication of QRPP towards QCPW, whereas for fifth graders, no relation is assessed.

If we compare the implicative network of the three graphs, we note that, for the three age-goups, the implication of the items PP towards PW is more frequent than the reciprocal implication PW towards PP. The pupils who succeed in Part/Part problems (PP) generally succeed in Part/Whole problems (PW), whereas the opposite phenomenon is less marked. For each task, the magnitude of direct implications PP towards PW increases with age (true indices respectively greater than .60, .78, .52 for QC, ignorance statement; greater than .71, .90 for QR and greater than .27, .72, .91 for OF). On the other hand, reciprocal implications PW towards PP remain comparable in size (in the same order, true indices greater than .44, .53, .39, ignorance .31, .21 and ignorance, .29, .25)

It is important to note that at the three ages, QR tests imply OF tests. The magnitude of these relations increases with age. Finally, for seventh and ninth graders, intermediate implications were found from OFPP to ORPW and from ORPM to QCPW. Furthermore, the implication from QRPP towards QFPW is intermediate for seventh graders (greater than .35) and increases for ninth graders (greater than .52).

No inductive conclusion of negligible implication could be drawn for the observed indices less than .20. Additional data would is required to draw any conclusion about them. However, an analysis carried out by pooling all age-groups together show that generally, non-drawn implications (less than .20) are negligible (most of the probabilities of the statement $\eta < .20$ are at least equal to .90).

Conclusions

Three main results emerge from this study. First of all, we can see that for fifth graders, a success in QCPW implies a failure in QRPP. When analysing the resolution process in details, we found that the pupils often use the same strategy for both problems ; this strategy, which was efficient for the first problem but irrelevent for the second one, was incorrectly generalized (Charron, 1996). The method presented here allowed to detect a conceptual obstacle for the young pupils that is overcome for older ones. Second, in order to succeed in PP type tasks, the success in the corresponding PW type task is necessary. Thus, the acquisition of the Part/Whole concept is necessary to the mastery of the Part/Part concept. Finally, tasks of QR and OF type share many relations of implication. These links suggest that they require common processes of resolution. The scope of these conclusions was controled through clinical interviews and qualitative analysis (Charron, 1995 and submited). More generally, we hypothesize that the near-implication relations found reveal operative invariants in construction.

Moreover, several suggestions concerning the teaching of mathematics can be outlined. The notion of Part/Whole relationship should be introduced to pupils before the Part/Part relationship. Furthermore, the fifth grade pupils should be warned that obstacles could appear at their level, so that they can avoid them. Both QR and OF tasks can be taught simultaneously while clarifying their common points and differences. Althought it is difficult to say that in every didactic situation, PW fractions are pre-required for PP fractions, we may netherthеless notice that this necessity is nowadays present in teaching.

The Bayesian Implicative Analysis allows to bring into light and to generalize the necessary stages for the acquisition of a concept. It was presented here from an example of study of the fraction conceptualization. It can be apply to other mathematical tasks and thus be useful to a better understanding of the development of mathematical competences. The AIB-2 software package, operating under Windows, allows all the computations and graphs presented here, and is available from the authors.

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HOW DOES ADDITION CONTRIBUTE TO THE CONSTRUCTION OF NATURAL NUMBERS IN 4- TO 13-YEAR-OLD CHILDREN?

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Classification and seriation are necessary conditions for number construction, but are they sufficient? The following types of exercises were given to 217 children aged 4 to 13: Discrete Quantities Conservation (DQC), Arithmetic Implications (AI), Number Classification (equality between quantities and between transformations), Ordering (between quantities and between transformations), and some addition problems. Success on DQC and AI implied success on some addition problems. Moreover, success on complex addition problems was accompanied by a change in numerical knowledge: natural numbers are considered as a particular class of signed integers.

Piaget and Szeminska (1941) thought that numbers were the result of the operational synthesis of classification and seriation, which occurs at the age of 7 or 8. Other studies have pointed out (1) counting abilities beginning at the preschool age (Gelman and Gallistel, 1978; Fuson and Hall, 1983) (2) later abilities related to mastering the sequence of counting words (Piaget et al., 1987). How can we explain this paradox? We could contend that the use of different criteria to detect abilities leads psychologists to study different areas of knowledge (counting tasks, conservation exercises, addition operations, etc.). We could also assume that the abilities needed to meet the same criteria change with age. For example, are the skills needed to succeed in conservation of discrete quantities the same at the ages of 8 and 12? Probably not.

But it is important in teaching number to know the real nature of the learner's knowledge, as well as the conditions under which it can be acquired. In this respect, an experimental study on the numerical sequence would be worthwhile (Vergnaud, 1991). Even if there is a great deal of research on number, few studies have dealt with the role of addition in number construction. In describing additive structures, Vergnaud (1991) showed that some preschool-age children are able to understand certain elementary properties, such as the fact that addition makes a quantity bigger and subtraction makes it smaller. He also showed that other properties have not yet been acquired in the teens. Could addition give numbers their properties of transformation, cardinality, and reversibility, and thereby contribute to the acquisition of conservation?

The aim of the present study was (1) to find out whether addition is also necessary to many stages of number acquisition, and (2) if so, to determine how additive structures contribute to the construction of numerical knowledge. For the remainder of this article, numbers referring to quantities will be considered as natural numbers (denoted N), the numbers referring to a relation, a difference, an addition, or a subtraction will be considered as signed integers (noted Z).

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Method

Population

The subjects were 217 children from six grades in twelve schools in the Parisian area: second-year preschool (mean age: 4 years 9 months), kindergarten (mean age: 5 years 10 months), first grade (mean age: 6 years 9 months), third grade (mean age: 9 years), fifth grade (mean age: 10 years 11 months), and seventh grade (mean age: 13 years 2 months). There were 31 children in each grade, except for the third grade where there were two groups of 31 subjects.

Procedure

The testing was done in two or three individual sessions lasting between 30 minutes and one hour depending on the child's attention span. The experimenter described the problem. The children answered orally and explained their thinking to the best of their ability. All children did the same four exercises to begin with, hereafter called the baseline tests, and then did exercises from one or two booklets (A, B, C) containing six problems each. The preschoolers were tested on exercise booklet A. The kindergartners and first graders, and the first group of third graders, were tested on booklets A and B. The second group of third graders, and the fifth and seventh graders were tested on booklets B and C. By construction, the exercises in booklet A were easier as a whole than the ones in booklet B, which were easier than the ones in booklet C. The testing order was counterbalanced.

The materials used were the same for each exercise (except for the baseline tests). They consisted of marbles of two different colors, boxes, and a grid to help the younger subjects count. Different numbers were used for each age: 0 to 4 for the preschoolers, 0 to 7 for the kindergartners, 7 to 15 for the first graders, and 9 to 30 for the third, fifth, and seventh graders. The exercises are described here with the values used for the third graders and above.

Exercises in the baseline tests

Discrete Quantity Conservation (DQC) (Piaget et al., 1941). A collection of 16 green marbles are lined up spaced at regular intervals. The child is asked to put out as many blue marbles as green ones. After the experimenter puts each blue marble next to a green marble for verification, the marbles in one of the rows are spread out and the child is asked, "Are there the same number of blue and green marbles?" (question 1: quotity conservation). Then the preceding manipulation is repeated and the child is asked, "Are there as many blue marbles as green ones?" (question 2: quantity conservation).

Arithmetic Implications (AI) (Piaget et al. 1987). The materials consist of two boxes located at different heights. The upper box (U) is connected to the lower box (L) by means of an opaque tube in which the marbles can drop from U to L. Two experimental situations are set up. In the first, the total number of marbles (T) is 15, 15 in U and 0 in L. In the anticipatory phase, the child is asked, "Before the twelfth marble drops down, how many marbles will there be in L and in U?" In the execution phase, the child makes the marbles go down. Then L and U are hidden and same questions are asked about the cardinals of U and L. In the second situation, there are 18 marbles in U and 0 in L. In the anticipatory phase, the child is asked, "If we drop 15 marbles from U to L, will the ninth one be in U or in L? And the tenth? And the fifteenth? And the sixteenth? And what other ones will be in U?" In the execution phase, the experimenter says, "You will drop 15 marbles into L." U and L are then hidden and the child is again asked the questions about the ordinals in each box.

Elevator (E). The subject is presented with a sheet of paper showing a vertical line of circles depicting the floors of a building, with as many basements as floors above ground. The middle circle represents the ground floor and is labelled zero. After having made sure the subject understands how the "elevator" works, the experimenter turns over the sheet of paper and asks the following questions. "The elevator starts at the ground floor and goes up 26 floors. What floor is it on?" (question 1). "The elevator leaves the 11th basement and goes up to the 13th floor, how many floors has it gone up?" (question 2). "The elevator moves up 24 floors from the 17th basement. What floor does it stop at?" (question 3). "The elevator goes up 23 floors and goes down 17 floors. Has it gone up more floors or gone down more floors?" (question 4). "How many?" (question 5). "And if it had started at the ground floor, where would it be? And what if it had started from another floor? Can you give an example?" (question 6). "The elevator goes down 17 floors, goes up 9 floors, and goes down 13, how many floors has it gone up or down?" (question 7).

Combinations of Opposing Transformations (COT). A box of 11 green marbles is presented and the child counts them (this will be called the initial stage, I_1, with $I_1 = 11$). The box is then hidden and 18 other green marbles which the child does not see are added. The box of marbles is presented again and the child counts (this is called the final stage, F_1, with $F_1 = 29$). The child is asked how many marbles have been added, which corresponds to the difference between F_1 and I_1 (question 1). The exercise is repeated with $I_2 = 28$ blue marbles and $F_2 = 15$ blue marbles (question 2). The child has F_1 and F_2 in front of his eyes and is then asked, "What have we done in all, have we added or taken away some marbles?" (question 3). "How many?" (question 4).

Booklet A exercises

Discrete Quantity Conservation (ADQC). The child is shown 16 marbles lined up in a row. First the child takes them in his/her hand. Then the experimenter lays them out in a square in the middle of the table, and finally moves them to the edge of the table. After each move of the marbles the child is asked, "Are there more, fewer, or the same number of marbles compared to before?" (questions 1, 2, and 3).

Equivalence Classes (AEC). The child is shown two boxes containing no marbles, two boxes containing 9 marbles, three boxes containing 12 marbles, one box containing 7 marbles, and three boxes containing 14 marbles. He/she is then asked to gather up the boxes which have the same number of marbles (question 1), justify his/her classification (question 2), and use a number-word to name each class constructed (question 3).

Equality Conservation in N (AECN). Two boxes of marbles of different colors but equal quantities ($E1 = E2 = 21$) are presented. The child has to count them (questions 1 and 2) and state whether there are more, fewer, or an equal number of marbles in each box (question 3). Then the experimenter points out the equality of two other collections of 9 marbles, which are added to the boxes. The content is then concealed and the seriation question is asked again (question 4).

Order Conservation in N (AOCN). The setup is comparable to AECN, but the two initial collections are unequal ($E1 = 17$ and $E2 = 18$). Questions 1, 2, 3, and 4 are the same as in the preceding exercise.

Searching for Part of a Whole (ASPW). A box is presented containing 14 blue marbles, which the child has to count (question 1). Then another closed box containing green marbles is presented and the experimenter says, "If I count the blue and green marbles hidden in this box, there are 27 marbles in all. How many green marbles are there in the closed box?" (question 2).

Commutativity in N (ACN). The child is shown 17 blue marbles and 19 green marbles and is told to count them (questions 1 and 2). He/she is then asked how many marbles he/she counted altogether (question 3). The experimenter then tells the child to imagine another child who did the addition in the other order and asks, "Did he find more, fewer, or an equal number of marbles compared to you?" (question 4).

Booklet B exercises

Associativity in N (BAN). There are three sets of marbles ($A = 13$, $B = 12$, $C = 16$). The child counts A and B and is told to perform the operation $A + B$ (question 1), count C, and then perform the operation $(A + B) + C$ (question 2). He/she is then told that another child did $A + (B + C)$, and is asked, "Did he find more, fewer, or an equal number of marbles compared to you?" (question 3).

Difference Conservation in N (BDCN). The child is shown 25 blue marbles and 11 green marbles and asked, "Where can you find the most?" (question 1) and "How many blue marbles were added? How many green marbles were taken away?" (question 2). The child is led to consider the equality between two other collections of 9 marbles, which are added to the two boxes. The content is hidden and the same questions are asked (questions 3 and 4).

For the following exercises, the same objects as for baseline test exercise COT were used. Adding will be called a positive transformation and subtracting will be called a negative transformation.

Order Conservation in Z (BOCZ). The child is told to count two unequal positive transformations (question 1 with $I_1 = 12$ and $F_1 = 25$, question 2 with $I_2 = 10$ and $F_2 = 24$) and asked, "To which box did we add more marbles?" (question 3). The child is then told that 12 unseen marbles were added on each side, and is asked the seriation question (question 4).

Equality Conservation in Z (BECZ). The setup is the same as above except that the two transformations are equal (question 1 with $I_1 = 12$ and $F_1 = 26$; question 2 with $I_2 = 9$ and $F_2 = 23$). Questions 3 and 4 are the same.

Combining Transformations and searching for a Negative transformation (BCTN). The child is told to calculate a positive transformation (question 1 with $I_1 = 9$ and $F_1 = 22$). Another box whose content is unknown is shown and then hidden, and some marbles are taken out. The child does not know the transformation, but is told that the result of the addition of both transformations is equal to +6. He must deduce that the unknown transformation is negative (question 2) and calculate it (question 3).

Commutativity in Z (BCZ). The child is told to calculate two positive transformations (question 1 with $I_1 = 11$ and $F_1 = 26$, question 2 with $I_2 = 14$ and $F_2 = 26$), and asked how many marbles were added altogether (question 3). Then the experimenter tells him/her to imagine that another child has done the addition in the other order. "Does he count more, fewer, or the same number of marbles compared to you?" (question 4).

Booklet C exercises

Associativity in Z (CAZ). The child has to calculate T1, T2, and T3 (question 1 with I_1 7 and F_1 26, question 2 with I_2 20 and F_2 4, question 3 with I_3 14 and F_3 27), and add three transformations (T1 + T2) + T3 (question 4). Then he/she is told that another child did T1 + (T2 + T3) and asked, "Did he find more, fewer, or an equal number of marbles compared to you?" (question 5).

Difference Conservation between Positive Transformations (CDCPT). The child calculates two positive transformations (question 1 with I_1 9 and F_1 24, question 2 with I_2 7 and F_2 23) and is asked, "Where did I add more marbles?" (question 3). "And how many more?" (question 4). He/she is then told (but does not see) that 11 marbles are added on each side, and is asked the same questions again (questions 5 and 6).

Order Conservation of Opposing Transformations (COCOT). The setup is the same as in BOCZ, but the two transformations are opposing (question 1 with I_1 9 and F_1 18, question 2 with I_2 26 and F_2 19). The child is asked, "Which box received the most marbles?" (question 3). He/she is told without seeing that 13 marbles are added on each side, and is asked the same question (question 4).

Difference in Negative Transformations (CDNT). The child calculates two negative transformations (question 1 with I_1 17 and F_1 9, question 2 with $I_2 = 25$ and $F_2 = 14$) and is then asked, "From which box did we take more marbles away?" (question 3) and "How many more?" (question 4).

Difference in Opposing Transformations (CDOT). The child calculates a negative transformation (question 1 with I_1 21 and F_1 12) and a positive transformation (question 2 with $I_2 = 11$ and F_2 25), and is then asked how many marbles were added in T1 compared to T2 (question 3).

Difference in Opposing Transformations, Searching for the Referrer (CDOTSR). The subject calculates a positive transformation (question 1 with I_1 13 and F_1 22). Another box of unknown content is presented, hidden, and some marbles are taken out. The child does not know what the transformation is but is told that the difference between the two transformations is +15. He/she must deduce that the unknown transformation is negative (question 2) and calculate it (question 3).

Results

The baseline test exercises DQC1, DQC2, AI, E and COT were scored as success or failure. The problems in exercise booklets A, B, and C were scored by grouping into exercise/ability pairs (table 1). An exercise/ability was considered to be performed well when two thirds of the questions in it were correctly solved.

Analysis of the baseline test scores indicated the following success order: DQC1, DQC2, AI, COT, and E. DQC1 were solved the best, and COT was solved the worst [t(21) = -

17 73***, prob(Lm/s< -1 153) = 90] The number of well-performed exercises in the baseline tests (from 0 to 5) increased with age [$Phi^2 = 0\ 97$, $p < .0001$]. In order to determine what notions are necessary to success on these exercises, the implication relations between the baseline test exercises and the items/abilities were noted. Remember that there is an implication (or quasi-implication $a \rightarrow b$), from exercise a to exercise b when, as a whole, subjects who exhibit a also exhibit b, while the reverse is not necessarily true. This relation can be quantified by Loevinger's H (1947). For values of H less than or equal to 0, the implication does not exist. When the index is 0, the two exercises are independent. The closer H is to 1, the higher the implication, and the implication is strict when H is 1. Table II presents the original matrix of observed implications between the baseline test exercises and the items/abilities in booklet A for the 4- to 9- year-olds.

Table I. Inventory of numerical abilities tested and exercises testing each

Numerical abilities tested	Name of exercises and question numbers involved
Counting (CG)	AOCN1,2 AECN1,2 ACN1,2 ASPW1
Equivalence class (EC)	AEC1,2,3
Discrete quantity conservation (DQ)	ADQC1,2,3
Equality in N (EN)	AECN3
Conservation of equality in N by translation (CEN)	AECN4
Equality in Z	BECZ3
Conservation of equality in Z by translation	BECZ4
Conservation of difference in N by translation	BDCN4
Conservation of difference in Z by translation	CDCPT6
Order in N (ON)	AOCN3 BDCN1
Conservation of order in N by translation (CON)	AOCN4 BDCN3
Order in Z	BOCZ3 CDCPT3 COCOT3 CDNT3
Conservation of order in Z by translation	BOCZ4 CDCPT5 COCOT4
Addition in N (AN)	ACN3 BAN1,2
Addition in Z	BCZ3 CAZ4
Searching for Part (Subtraction) (SP)	ASPW2
Difference in N (Subtraction)	BDCN2
Difference in Z (Subtraction)	CDCPT4 CDNT4 CDOT3
Calculation of a positive transformation	BOCZ1,2 BECZ1,2 BCZ1,2 BCTN1 CAZ1,3 CDCPT1,2 COCOT1 CDOT2 CDOTSR1
Calculation of a negative transformation	CAZ2 COCOT2 CDNT1,2 CDOT1
Negative sign and calculation of a negative transformation	BCTN2,3
Negative sign and calculation of a negative transformation of a referrer	CDOTSR2,3
Commutativity in N (CN)	ACN4
Commutativity in Z	BCZ4
Associativity in N	BAN3
Associativity in Z	CAZ5

Table II. Original matrix of observed implications between the baseline test exercises and the items/abilities in booklet A

	Count	Equivalence Classes				Order		Additive Structure		
$a \rightarrow b$	CG	EC	DQ	EN	CEN	ON	CON	AN	SP	CN
DQC1	138	311	459	507	**.775**	261	**.534**	377	384	332
DQC2	1	1	**.878**	1	1	1	**.916**	**.802**	**.778**	**.683**
AI	1	1	1	1	1	1	**.664**	**.604**	**.778**	**.62**
E	1	1	1	1	1	1	1	1	1	1
COT	1	1	1	1	1	1	1	1	1	1

Looking at the boldface characters in table II, we can see that success on DQC (1 and 2) strongly implies abilities in order, equality, and additions in N. Concerning COT, E and AI, the value of the index is close to or equal to 1. An identical analysis performed on booklet B abilities (for 5- to 13-year-olds) and booklet C abilities (for the 9- to 13-year-olds) showed that the conservation of difference in N or in Z implies conservation of order, conservation of equality, and the calculation of transformations.

In order to explain the role of addition in the construction of numbers, the correct answer patterns for exercises involving the conservation of equality, order, and difference, in N and in Z, will be discussed. At the lowest level, the child answers simply, sometimes without explanation, as in "There are as many marbles as before" (for AECN4). Other children can explain their answer, as in "There are more green marbles added because you put in six green marbles and four blue marbles a minute ago, and now five and five (first grader for BOCZ4). At the next level, the child explains a general rule, saying that adding the same number to two collections does not change the preexisting relation (of order or equality) between the two, as in "You added the same number of marbles, so it is like it was at the beginning, it doesn't change anything" (fifth grader for BDCN4). At the highest level, the child understands order conservation as an invariant that is independent of addition, as in "It is just like multiplication, both results are high and it doesn't change the final result because you added nine here and nine there. The two results get higher in a same proportion" (third grader for BDCN4).

The same pattern was found for commutativity in N (ACN4) and in Z (BCZ4), and for associativity in N (BAN3) and in Z (CAZ5). The child interchanges the boxes without explanation. During verbalization, the child agrees that A + B = B + A, as in "It's the same because it means the same thing; it's the same addition, 12 + 15 and 15 + 12" (third grader for BCZ4). This phenomenon is seen as a result of adding. Then this result is abstracted and made into a general rule, which becomes a property of addition, as in "When we add, whether the number is before or after makes no difference." At a higher level, the seventh graders accurately name the tested relation as an independent entity, as in "It's commutativity, "It's associativity", or "This is found in multiplication too."

As for transformations, which were part of various exercises, they are regarded as the addition or subtraction of a natural number, as in "You added 9" (all ages). Later on, these transformations become objects referred to by a name (e.g. the difference), as in "Because 14 is positive and 9 is negative, the difference is 14 + 9 = 23" (for CDOT3). Others consider this entity as a signed integer with all of its properties, as in "They are signed integers, so we can add them" (seventh grader for COT). At this level, the natural number is no longer a quantity. It also becomes a particular kind of signed integer.

Conclusion

As expected, two main results stand out from this study. (1) Addition, like equality and order, contribute to the development of numbers from preschool age on. (2) Natural numbers gradually acquire more and more, increasingly complex properties based on order and equality relations and additive structures. One of these properties, the quantification of differences (additions or subtractions), is constructed like a signed integer. The latter leads the subject to reconsider natural numbers, which then become positive signed integers.

At first, concepts are problem-solving tools, whether implicit or explicit. They are initially seen as the properties of another existing mathematical object, such as addition. Some of these tools are then granted the status of an object, with properties of its own. The shift from the tool to the object leads to the re-elaboration of former knowledge. The use of a variety of addition problems as early as preschool can contribute to reinforcing the acquisition of number.

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OVERCOMING ISOLATION IN TEACHING: A CASE STUDY OF COLLABORATION IN A PROFESSIONAL DEVELOPMENT SCHOOL MATHEMATICS DEPARTMENT

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This paper analyzes interviews with teachers who shared teaching assignments in which they explored an innovation in the teaching of lower track Algebra One classes. The teachers emphasize two important ways in which sharing a teaching assignment lead to their professional growth. They emphasize the value of the opportunity to experience an innovation in context and the importance of having to make joint pedagogical decisions.

Sociological descriptions of teaching (e.g., Lortie, 1975) and analyses of the culture of teaching (e.g., Feiman-Nemser & Floden, 1986) portray an occupation in which practitioners operate individually behind closed doors. In attempting to explain this phenomenon, Huberman (1993) suggests that the development of an instructional repertoire is an inherently idosyncratic process; teachers are artisans. Yet, research suggests that teacher-teacher professional collegiality can serve as an important catalyst for professional development and school change (see Griffin, 1991 for an elaboration of this argument). Such teacher collegiality can take many forms from occasional conversations about teaching; to the sharing of lessons, materials, and classroom stories; to intensive collaboration on a shared teaching assignment.

Proponents of collaboration among teachers suggest that "what is most deeply known *about teaching* is known by *teachers*" (Griffin, 1991, p. 250) and that therefore it is sensible to have teachers learn from each other. Some (e.g. Rosenholtz, 1989) argue that this learning should happen in the classroom. They focus on organizational arrangements in which teachers learn from each other by sharing responsibility for the instruction of a group of students (see the review by Cohen, 1976). Because of his emphasis on teacher artisanship, Huberman is skeptical. He argues that, "it is hard to imagine two such people equally responsible for the same pupils at the same time. The response set of one person would collide, early on, with that of the second, whose reading of the situation and whose rapid, on-line response would necesarily be different..." (1993, p. 17-18).

This paper presents a case study of one high school mathematics department's attempts to develop professionally through shared teaching assignments whose goal was to create an innovative Algebra One experience for students in lower track classes (referred to as "team teaching" below). Though, as Huberman suggests, there were tensions inherent in the sharing of a teaching assignment, the teachers reports of their experiences were quite positive. In this paper, we analyze interviews with the teachers with our focus on the explanations given for their perception that shared teaching assignments present unique professional development opportunities.

Background/Context

Holt High School, the setting for the events described in this paper, is located in a suburban/rural community ten miles south of Lansing, the capital of Michigan, and a similar distance from East Lansing, home of Michigan State University. Historically, there were always connections between people associated with Holt High School and others associated with Michigan State, as there are with many local K-12 schools. But, the nature of these connections and the level of institutional involvement escalated dramatically in the mid-eighties. In the mid-eighties, Michigan State University revamped its teacher education program and began to implement Holmes Group notions of professional development schools (see Holmes Group, 1986). The notion was to create institutional connections with schools like Holt High School around teacher education, the development of exemplary practice, and educational research. The vision suggested a synergistic set of activities and collaborations between university faculty and K-12 educators around important issues of educational practice.

Michigan State University's Professional Deveopment School initiative at Holt High School is a complex one which defies simple description. On one level, it has meant the infusion of resources into the high school. Within the Mathematics Department, these resources led to a series of projects. We will concentrate on one strand in the work; concerns about students in the lower track classes of the high school which led to changes in the lower track curriculum -- the gradual abolishing of pre-Algebra, General Math, and Practical Math -- and to a focus on the teaching of new approach to Algebra One. The Algebra One work has been the locus for team teaching. For three years, a member of the math department taught with a university faculty member. During the 1993-1994 year, the department decided to use PDS funds to allow two department members to teach together. During the 1994-1995 school year, each of these teachers taught Algebra One with another member of the department.

Algebra One Team Teaching in 1994-1995

Relevant literature (e.g., Geen, 1985) suggests that it important that team teaching be organized around some defining purpose, be initiated by the involved teachers, and be supported by administrators and/or department heads. Many of these conditions were met at Holt High School.

The Algebra One team teaching at Holt centered on a new approach to algebra, a subject which is often problematic for students. It focused on a change in how the x's of algebra are conceived. In this approach, instead of viewing x's as unknown numbers, x's are treated as variables which can take on a range of numerical values (See Bethell, Chazan, Hodges, & Schnepp, 1995 for a development of this approach; Chazan, in press). This change of emphasis has lead the department to develop a set of materials and activities designed to promote classroom discussions and to replace textbook presentations. But, these materials do not constitute a linear curriculum. They pose open-ended questions, suggest the examination of calculation procedures used in day to day life, involve alternative assesement, and make use of graphing calculators and computers. They are under continual development and can be used in different orders; some teachers have particular preferences in ordering the materials and others try to shape their instruction in response to comments their students make.

Based on previous work, for the 1994-1995 school year, the mathematics department wrote a Professional Development School proposal calling for the participation of four teachers (half of the math department) in two Algebra One team teaching pairs. Each pair agreed to share year-long responsbility for teaching one Algebra One section. Both teachers would be present each day and would share the instructional tasks. But, though the work was joint, the teachers contributions to the collaboration were not identical. Each pair included a teacher experienced in the curriculum and a teacher new to the curriculum. In each of the two pairs, the teachers who had team taught Algebra One during the previous year agreed to help introduce the other team member to the new approach to algebra. They agreed to provide, especially early on, the big picture and direction as well as ideas about the use of previously developed curricula material.

However, the teachers who had experience with the curriculum were also interested in learning from their colleagues. They each hoped to participate in the further development of the curricular materials and to grow in their personal understanding of teaching algebra. Since all four members of the two pairs were experienced teachers, ranging from 7 years of experience to more than 20 years of experience (one of the teachers not experienced with the curriculum was the department chair), they expected to learn from the particular expertise of their colleagues (e.g., with alternative assessment) and from their general wisdom accumulated during years of teaching experience.

The study

The fundamental data collected in this study were transcripts of structured interviews carried out with all four teachers. Each teacher was interviewed in the fall and in the spring. Each interview was carried out by a team of two interviewers, lasted approximately ninety minutes, and was audiotaped. The interviews were structured to collect data about the teacher's changing views of algebra and their experience of team teaching.

The data were supplemented by visits to the Algebra One classrooms, notes taken during weekly teachers meetings, and by interviews with a small sample of students in the Algebra One classes.

Over the course of the year, though there were numerous examples of tensions in these relationships, the teachers' informal assessment of team teaching emphasized the unique and perhaps even transformative opportunities which it represented. Therefore, the analysis focused on their reasons for feeling that the opportunities presented by team teaching were unique.

Focused Analysis of Teacher Interviews

We classified the reasons given by the teachers for ascribing a special value to the team teaching under two broad categories. First, the teachers saw great benefit in their being physically present in the class with another teacher. They felt that this shared experience allowed for the construction of mutual understandings which often could not be captured or conveyed in words. Second, by teaching the same material to the same students at the same time, the teachers had unparalleled opportunities to collaborate in the midst of the complexities of teaching -- from attempts to assess individual student understanding to decisions about course content and sequencing of the material for the whole class. The shared teaching assignment forced teachers eventually to take action. They had come to an agreement or compromise. These two aspects of the team teaching form the framework for our analysis of the interviews.

Not *having* to rely on words: Shared experience of the same classroom instead of talk about the classroom.

The teachers interviewed often compared conversations resulting from team teaching favorably with departmental conversations about classes taught individually. Their descriptions highlight the difficulties inherent in having discussions about teaching without opportunities for shared examination of the teaching itself. The following quote from Mike is in response to a follow up question to a statement of his in which he claimed that team teaching had more impact on his understanding of the algebra curriculum than the stories and sharing he had participated in during the previous 3-4 years. We asked him why he thought team teaching had this impact.

> I had a picture in my mind about what the classroom might look like [when hearing stories about instruction], but it was my classroom. It was my set of norms, the way I operate a classroom. Team teaching helped me understand how Sandy set up the norms of the class and the time set up setting up the norms. But also how she questions kids in such a way that is not threatening to them, so it's made me more aware of how I question them. (Mike, Fall 1994)

As stated earlier, one of the goals for these team teaching situations was to work with different curricular ideas. Rather than following a textbook page by page, these teachers were working on developing an approach to algebra that presented students with rich problems, allowing students the opportunity to analyze and extract the pertinent mathematics, as guided by the teacher. In this context, the curriculum was a living phenomena, rather than words or ideas taken out of the natural environment of the classroom. Teaming allowed teachers to experience the curriculum in context and not in an out of context verbal description. This phenomena is described by Marty in the following two quotes:

> I heard Sandy talk about it [changes in curriculum], I looked through the problems and stuff, but without seeing how it played out in the classroom, I couldn't really understand the differences. And I had a lot more concerns - that the kids would be lacking certain things when they came out [of the class], because [when you look at the problems] you miss the entire discussions part of any of the problems, which is the key to when the mathematics starts coming out...I was talking to one of our special ed. teachers who was at an elementary school and we were talking about teaching math differently and she said she talked about it and talked about it and talked about it and didn't get it until she team taught it. And I thought, ... that is the conclusion we are coming to. (Fall 1994)

Building on the shared experience, the teachers then claimed to be able to construct a different sort of vocabulary to talk about teaching and to have different kinds of conversations. They claimed that they were able to use a shared vision of the class, shared understanding of class norms, and shared observations of students' interaction with the material as a backdrop for their discussions. Marty (Spring 1995) said, "The thing that I see is that it [team teaching] gives us a common experience in mathematics so that we can have a common language to talk about it."

The idea that you need to witness the students doing mathematics differently was furthered emphasized by Marty:

> Until you see what goes on in the classroom ... I don't think a person would understand otherwise. That's really pretty limited information if you are just looking at test scores or whatever. It's the change in attitude that the kids have in the classroom about math. It's what they're doing in the classroom and how they understand the material, not just if they can do problems with it. I think without team teaching we're not going to get that point across to people, until they can actually see it and talk about it with somebody as it's happening. (Fall 1994)

In discussing the students' progress and understanding, Steve, who had found verbal descriptions of the class hard to understand, also mentioned the importance of what he had actually witnessed. Steve was in the unique position of teaching an algebra class based on a textbook during the same time as he was team teaching with Marty with the new approach. His experience lead him to make comparisons:

> Probably the biggest thing that I carry away from the class is the fact that I think the kids generally have a better feeling about math than a textbook driven class, just possibly because their investment in it would be in the discussions. Usually, we have some really good discussions with the class. Yesterday was an interesting one where kids were very eager to volunteer. (Spring 1995)

Having to decide together:Joint reflection, decision making, and action instead of individual responsibility

But, the teachers were not sharing an experience in which they were passive observers; they had responsibilities to act in concert. In an article entitled "The Persistence of Privacy: Autonomy and Initiative in Teachers' Professional Relations," (1990), Judith Warren Little presents a possible continuum of collegial relations (p. 512). This continuum moves from more independent teacher relationships to more interdependent ones. The team teaching in Holt High School's Mathematics Department shares many characteristics with the kind of collaboration that Little (p. 519) refers to as "joint work." Joint work is an interdependent relationship between teachers in which the teachers have a shared responsibility for the work of teaching, collective conceptions of autonomy, support for teachers' initiatives, leadership with regard to professional practice, and group affiliations grounded in professional work. It is rare at the secondary level. In describing the difference between the independent and interdependent interactions between teachers, Little suggests:

> Collegial relations that center around storytelling, mutual assistance, or sharing issue slight challenge to autonomy conceived as personal prerogative. Teachers in productive teams, departments, groups, and projects express an alternative conception. The demands and the prerogatives of professional autonomy shift from private to public, from individual to collective. ... Teachers open their intentions and practices to public examination, but in turn are credited for their knowledge, skill, and judgment. (p. 521)

The teachers at Holt had much to say about the ways in which team teaching provided opportunities for, and sometimes even forced, collaboration and public discussion of teaching. Though some aspects of collaboration -- even when ultimately rewarding -- could be uncomfortable, others were described as pleasurable. For example, as the department chair, Mike commented on the difference in the conversations when you are teaming versus when you are holding department meetings.

> But somehow, I really think that teaming *forces* you to talk about the issues that in meetings we dance around all the time. Somewhere along the line you have to talk about homework, you have to talk about content, how you present content - what is important, what is not important in the curriculum. You don't have a choice. (Fall 1994-5, emphasis added)

Sandy (Spring 1995) commented on the shared and continual sorting out of what is going on in the class. She claimed that team teaching forces one to articulate rationales for everything you do with students -- from disciplinary action to grading to curricular choices. For example, Steve and Marty collaborated closely on the grading of papers, consulting each other about the answers to open-ended assessment questions and reaching a consensus on credit.

In contrast to the sometimes uncomfortable necessity to articulate one's practice, one aspect of this collaboration which was particularly enjoyable to the teachers was its emphasis on mathematics. Since the focus of the team teaching was on new approaches to algebra, coupled with a pedagogy that allowed students to construct knowledge and develop their own understanding, teachers were confronted with basic questions about the meaning of algebra, teaching, and learning. The teachers found this aspect personally enriching. Marty commented on this aspect in the fall and the spring:

> I think that's one of the best things about team teaching, you're getting together with other math teachers and just being able to talk about math and talk about new ideas. (Fall 1994)
>
> A lot of times where we're working after school we'll start putting a problem together an we'll get a little carried away and add a few more because it's fun. That's one of the neat things about the team teaching - when you're planning together, if you can find the time in our schedules. It's really enjoyable and you can get creative with problems and I think that really shows in the classroom, too. The kids pick up on that - if you come up with a problem that they think is really interesting and unique. I think that happens more often when you're working together with someone that if you're working alone. (Spring 1995)

At other points in the interviews the teachers talked about the opportunity to take risks that one wouldn't take alone. They talked about the added confidence with which one can try new ideas or teaching techniques as a result of having the support of a colleague. Mike saw these benefit as a result of the development of trust and of familiarity with another person's thinking. As department chair, he thought that such trust and familiarity might aid in the development of wider collegial relations in the department.

> So I think the more we can team the more we can face the issues. Also, I think the more we can team, the more we can develop a real trust of each other. I mean, you have to trust the person you're teaming with, otherwise they're going to make you look like an idiot in front of thirty-one kids. So you learn to trust them. You learn how they think and why they do. So when you have these conversations, you have a better idea where they're coming from. So their ideas aren't off the wall in your opinion. Because they actually do think about things. (Fall 1994)

Conclusion

This analysis has focused on two sets of reasons given by teachers for finding a shared teaching assignment professionally rewarding. According to the teachers, the team teaching within the Mathematics Department at Holt High School has produced important changes in teacher relationships, professional development, curriculum development, and instructional practices. They claim to have found themselves thinking differently about the subject matter, their colleagues, students and themselves. Will these reported changes affect their practice and the tenor of their departmental and professional interactions in the future? Are the changes evident in classes these same teachers teach by themselves? Will the work in Algebra One spread into other areas of the curriculum? These are some of the questions we intend to continue to pursue.

At the same time, we have other questions, ones about the range of applicability of team teaching as a professional development tool. Clearly, compared to traditional inservice activity, team teaching is a slow and expensive method of professional development. Nonetheless, it does seem promising; the teachers' self-report is positive; they report overcoming isolation and learning from each other. In the future, we hope to explore whether there are other ways for teachers to share experience and to make decisions collaboratively which perhaps might be more widely applicable.

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NESB MIGRANT STUDENTS STUDYING MATHEMATICS: VIETNAMESE AND ITALIAN STUDENTS IN MELBOURNE[1]

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This paper describes one part of a longitudinal project studying bilingual Arabic, Vietnamese and Italian migrant children learning grade 4 mathematics in Melbourne and Sydney, Australia. Although the project has drawn on the work of Cummins (1979, 1991), we are also particularly interested in why students switch between their languages when processing mathematical problems. What may prompt a bilingual student to switch languages? How often does it occur? Does it depend on the mathematical context? What changes might occur as the student progresses through the primary school? This paper will comment on initial findings from the first phase of data.

Although Australia's migrant intake has remained reasonably stable at about 100 000 per year for the last 20 years, there has been a change in the origins of immigrants. In the late 1960s well over half our immigrants were from English speaking countries. This figure has now fallen to below 25% (Bureau of Immigration Research, 1992). The 1988 Census reported that 2 220 000 or 14% of Australia's 16 million people spoke a language other than English in their home. Such changes in the immigration profile have in turn led to a major increase in the number of Non English Speaking Background (NESB) students in schools [NESB people who were born or their parents were born in a non English speaking country].

Up until the early 1970s it was assumed by most educationists that being bilingual bestowed no advantage for school learning. However since then data has started to accumulate which indicates that in particular situations bilingualism can be an advantage. Much of this has come about by a clearer understanding of what is meant by the term bilingualism (for example see Cummins, 1979, 1991).

During the last twenty years the links between language competence and mathematics learning have been a major area of research for mathematics educators (see Ellerton & Clarkson, 1992). The broad pattern of this interaction is now starting to emerge for monolingual speakers. Bilingualism was recognized early in this movement as an important component, but has for various reasons not held center stage. Few studies have taken up the specific issue of mathematical competence of NESB students in Australia. The two major ones concentrated on NESB children's performance compared to monolingual students (Ainley, Goldman & Reed, 1990; Hewitt, 1977).

This project will report on the effect of level of competency of languages on mathematical performance at a later time. In this paper, preliminary results concerning Vietnamese and Italian students' behavior of switching between languages when completing mathematical items will be reported. Earlier research elsewhere noted that bilingual students regularly switched languages when attempting to solve mathematical problems (Clarkson, 1992; Dawe, 1983). Although it was not possible to follow this matter up in any depth in those projects, it was surmised that this could be an important factor in a bilingual student's attempt at processing mathematical problems. We would also note that although there have been many studies and much speculation on the role of code switching in general (for a number of studies in various settings see Bialystok, 1991), there seems to be none in relation to mathematics learning.

The first question to ask is, do these NESB students in an Australian context switch languages when attempting mathematical questions, and if so with what frequency? Other questions follow: Do students who respond in this way to mathematical questions make more errors? What prompts students to switch languages? Do students simply have a preferred language in which they wish to work when it comes to mathematics? A further possibility deals with the perceived difficulty of a problem. Does this prompt students to switch languages? Another interesting factor is that of the mathematical context. Will there be a variation if students are confronted with symbolic algorithmic items, with routine mathematical word problems, or with open ended mathematical questions? The answers will have important implications for teaching and learning mathematics.

Methods

Subjects

Overall, four groups of year 4 students are involved in this study. Experimental target groups are NESB Vietnamese, Arabic and Italian students. A monolingual English speaking group of students form a comparative group. Schools were chosen on two criteria. The first was that they had a positive commitment to bilingual students and the use of their L1, and the second was the ease of access by the researchers into the school. All students included in the study will have completed all their schooling in Australia. This stipulation is included since some immigrant students who have completed little schooling in their home country are sometimes started in higher grades when admitted to an Australian school because of considerations to do with age. Such students may well introduce extraneous effects. Year 4 students have be chosen since a longitudinal format for the study will be followed. It is planned to visit students again when they are in years 5 and 6.

In this paper we give a preliminary report on the Vietnamese and Italian students who live in Melbourne. Although we targeted 100 Vietnamese students who we were advised by their schools were NESB, an examination of the student data reduced this to a group of 80. It eventuated that some students had only one parent who was born in Vietnam, and some students spoke Chinese rather than Vietnamese as their L1. With the Italian students 95 were nominated by schools, but only 32 met our criteria when data was examined. This calls into question just how well schools do know the background of their students. The students were drawn from three government and eight Catholic schools. The schools were all suburban, with the majority of families living generally in working class or middle class areas. The amount of school support for the bilingual students varied from bilingual teachers being on staff and running combined mother tongue/LOTE programs for up to 2 hours per week, through to bilingual aides being available when needed.

Instruments

Mathematical Tests Three formats were used:

(i) Symbols Test: This test consists of items 1 to 10 and 21 to 30 of the ACER Operations Test (Australian Council for Educational Research, 1978) giving a maximum score of 20. All items are composed of mathematical symbols with no words. For all items there is a stem followed by 4 or 5 alternative answers given.

(ii) Maths Word Problem Test: A range of 10 mathematical word problems drawn from the ACER PATMATH series 1 (Australian Council for Educational Research, 1988), and hence a maximum score of 10. The items cover number, measurement and spatial topics. The format of each item is a stem with 4 or 5 alternatives.

(iii) Maths Novel Problem Test: 10 'open' items of the type that Sullivan and Clarke (1992) have explored. Such items do not have one right answer and hence may well present the students in this study with mathematical situations that are quite novel to them. Items are of an extended answer format. Two scores were computed from this test. The first was the 'raw score' which was calculated by allocating one mark for each item where there was at least one correct answer given, giving a maximum score of 10. A second score was calculated by allocating to each item a score of 1 if one answer given was correct, a score of 2 if two answers were correct, and 3 if three or more answers were correct. Incorrect answers, when multiple answers were given, were ignored. Hence the 'novel score' could range between 0 and 30.

Language Information Sheet. An additional sheet was attached to the back of each mathematics test instrument which asked for information on which language was used to process each item. The first column listed the number of the items of the test. Five more columns were headed 'used all English' through 'used part English and part Vietnamese (or Italian)' to 'used all Vietnamese'. After completing the test students were asked as a group to turn to this page. After some discussion about how monolingual people can only think in one language, but people who know two languages might think in one or the other, or sometimes swap backwards and forwards between them, students were asked to look at this sheet. The supervisor then said that s/he would like to know about the language(s) that students thought in when completing the test. The students were asked to look back at item 1 and remember as best they could which language(s) they used to think about it. They were then asked to tick the appropriate column. After completing the second item as a group the students were asked to work through the other items by themselves, each time looking back at the item before ticking a column. The supervisor then moved around the group (with the classroom teacher if available) handling any queries and ensuring that students were not confused. In each class there were some students who need further discussion about what to do, but all students completed the task adequately. For analysis purposes the instrument was treated in a similar way to a Likert scale with 5 being allocated to 'all English' through to 1 for 'all Vietnamese'.

Interview Schedule Selected NESB students were interviewed in a 1-1 situation. The interview was video taped. To begin the interview, students were asked to solve three or four unseen mathematical problems, some of which were of the 'novel' variety as described above. The video of them attempting the solution of these problems was then replayed and used as a stimulus in discussing how they had gone about the solution process, and in particular whether and at what stages they had switched languages if they had indeed done so. During the interview the student was also asked about their use of L1 when doing mathematics in the classroom, and outside of school, whether they attended special language school at weekends, and who if anyone helped them complete their maths homework.

Results and Discussion

Occurrence of Switching

It was suggested above that key beginning questions to this investigation were the following: Did students switch languages when attempting the mathematical items? If it did occur, how frequent was it?

To gain some insight into these questions, the number of items for which students reported using their first language was calculated. These results are presented in Table 1. It should also be noted that only 2% of Vietnamese students and 3% of Italian students indicated that they used their L1 for all items on all tests.

TABLE 1: Use of L1 in solving maths items (Vietnamese N=80; Italian N=32).

Math Tests	**% of students using L1 for at least one item.**		**% of students using L1 for all items.**		**% of students using L1 for no items.**	
	Vietnamese	**Italian**	**Vietnamese**	**Italian**	**Vietnamese**	**Italian**
Symbols	43	22	14	9	56	78
Word Problems	53	28	9	10	47	73
Novel Problems	41	28	8	9	59	73

It is clear from Table 1 that there is a large minority of Vietnamese students who regularly use their L1 in solving at least some mathematical problems. The proportion of Italian students using their L1 is about half that of the Vietnamese students. Since many more of the Italian students were first generation Australians, this difference is expected. In fact these percentages of Italian students surprised most of their teachers.

An examination of a percentage frequency table for each maths test (Tables 2 and 3) is similar for both language groups in that there is little clustering of students who use their L1 to complete a few items, or who prefer to complete nearly all items using L1. Rather the data tends to indicate that the students are reasonably spread out. The one partial exception to this may be for the Symbols test which does seem to have more students than the other two tests electing to use their L1 for all items. However this effect is probably explained by the fact that this test has twice the number of items than the other.

Difficulty

The discussion of results will now focus on those Vietnamese students who reported using their L1 to think about the solution for at least some of the mathematical items. There were not enough Italians in this category to calculate comparable statistics.

TABLE 2: Number of items on the Symbols Test versus the percentage of students who chose to use L1(Vietnamese N=80; Italian N=32).

Number of items for which L1 used	% of students Vietnamese	Italian	Number of items for which L1 used	% of students Vietnamese	Italian
0	57	78	11	3	0
1	0	0	12	1	0
2	5	0	13	1	0
3	1	3	14	3	0
4	3	0	15	1	0
5	1	0	16	0	0
6	0	0	17	0	0
7	1	0	18	1	3
8	0	0	19	3	3
9	1	0	20	14	9
10	3	3			

TABLE 3: Number of items on the Word and Novel Problems Tests versus the percentage of students who chose to use L1 (Vietnamese N=80; Italian N=32).

Word Problem Test			Novel Problem Test		
Number of items for which L1 used	% of students Vietnamese	Italian	Number of items for which L1 used	% of students Vietnamese	Italian
0	47	73	0	59	73
1	6	7	1	4	7
2	6	3	2	5	3
3	5	0	3	1	0
4	6	0	4	7	3
5	5	3	5	3	3
6	4	0	6	3	0
7	3	3	7	4	0
8	10	0	8	4	0
9	0	0	9	1	3
10	9	10	10	8	7

It was suggested that difficulty may be associated with the use of L1. First of all we focused on the students. We did this by comparing the Vietnamese students' test scores with the number of items for which they used their L1 in the solution process. Table 4 shows the correlation coefficients calculated. It would appear that for the Symbols and Word Problem Tests, there is little association between the score gained on the test and the number of times students used Vietnamese. However for the Novel Problems Test there is a small but significant result. These figures suggest that students who obtained a higher raw score and a higher 'novel score' on this test, also completed more items using Vietnamese.

In focusing on the items rather than the students, we looked for an association between the item difficulty and the number of students who completed that item using Vietnamese. The resulting correlations are shown in Table 5. The results in Table 5 suggest little association between item difficulty and the number of students who used their Vietnamese to complete the item for the Symbols and Novel Problem Tests. However for the Word Problem Test it would appear that with an increase in item difficulty there was also an increase in the number of students who used Vietnamese.

TABLE 4: Correlations between score on tests and number of items for which Vietnamese was used.

Mathematics Tests	N	Correlations
Symbols Test	30	-0.13
Word Problem Test	41	-0.14
Novel Problem Test	30	
raw score		0.37*
novel score		0.36*

* sig at the 0.05 level

Table 5: Correlations between the item difficulties and the number of students who completed the item using Vietnamese.

Mathematics Tests	N	Correlations
Symbols Test	19	-0.08
Word Problem Test	27	0.67*
Novel Problem Test (raw sc.)	28	-0.11

* sig at the 0.05 level

At this point it would be premature to read to much into these results. However it does suggest that difficulty may well be an important variable to follow up in further analysis of the full results. This was reinforced by some of the statements we had from students during the interviews that were conducted.

Student 1:

St1: Well I plussed because I did 6 plus 6 is 12 and I plus 4 more.
P: So you got 16. Did you swap back into Vietnamese at all or do that in English?
St1: English
P: Was that because this one you found a lot easier?
St1: Yes
P: Is it when it gets difficult you swap into Vietnamese?
St1: Sometimes when it is difficult I go back in my mind to Vietnamese and then I do it.
P: Good well that was finished there.

Student 2:

St2: 25 ...there must be more than 10, but then...
P: I see ...Did you do all that calculation in English when you were adding and subtracting .. or did you use Vietnamese at all?
St2: English
P: OK
St2: Sometimes if it is hard I do in Vietnamese

P: But this one was just confusing . ? Do you use Vietnamese much doing classwork..
St2: Sometimes
P: When?
St2: When it's hard.

Student 3.
St3: I just said 4+6
P: Straightaway. . Did you do that in English or Vietnamese?
St3: Vietnamese
P: Do you do all your numbers in Vietnamese?
St3: Yes
P: and why do you choose Vietnamese?
St3: It was sort of like easier.
P: OK and you......
(Similar excerpts could be quoted from various Italian students.)

However not all students reported that the perceived difficulty or their actual inability to progress with a solution prompted them to switch languages. Hence this is only one, but probably an important factor, in explaining why students do switch languages. It was also noted from the interview transcripts that students may well choose to complete aspects of the solution process in one language and use their other language to deal with different processes. Hence some children would read or reread in one language but switch when it came to processing numbers. One or two did the opposite to this.

Context

The reason for using three different mathematics instruments was to explore the effect of context. It has already been noted that for all tests there were a number of students who chose to use their L1 for at least some items. Hence from that perspective, context does not seem to matter. However the differing results for the correlations calculated does suggest that there may be an effect here worth exploring.

Teachers' Comments

When we started working with schools there was a range of incidental comment from the classroom teachers. Some of the more interesting concerned their perceptions of whether children in their class would use L1 while doing mathematics. On the whole many teachers were not convinced that this would be so. All were aware that the Vietnamese children did use their L1 to varying degrees to speak to their peers in the playground. A number of these students also used Vietnamese in class, but for general clarification of the classroom procedures and in discussing work more in the humanities area. Very few of the teachers who taught the Italian students even considered the possibility that these first generation Australian children would use Italian at all during school time, although there might be some use in the home. This outlook is in line with survey results that were conducted some time ago where teachers saw little reason at all for using L1 in mathematics and science classes (Clarkson, 1993). Such attitudes are surprising given the highly multicultural profile that Melbourne has. The results from this study may help teachers to focus on the fact that bilingual children may well switch languages for all areas of their class work, as well as for talking to their friends outside of school.

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LEVELS OF FORMAL REASONING IN SCHOOL ALGEBRA

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This paper explores higher-level skills associated with school algebra. In particular, it considers three topic areas that are problematic for many students in the upper-secondary school. Three questions are discussed that have been taken from a pool of questions used to explore students' approaches to aspects of variable substitution, function notation and algebraic fractions. These questions were chosen as they were identified as non-routine by the 150 tertiary students in the sample, and required the utilisation of more advanced skills. The results indicate similar response patterns, covering all topic areas.

Introduction

Fundamental to the notion of formal reasoning in algebra is students' ability to understand the structures underlying mathematical concepts so that the identification and subsequent analysis of relationships can take place. It is this very aspect that forms the basic premise on which the SOLO Taxonomy's description of the formal mode is based. According to Collis and Romberg (1991), "The elements are abstract concepts and propositions, and the operational aspect is concerned with determining the actual and deduced relationships between them; neither the elements nor the operations need a real-world referent" (p. 90). Collis, Romberg and Jurdak (1986) note also that " ... the structure of the learned responses ... becomes increasingly more complex" (p. 207).

In this mode, students begin to question why things are as they are, and can support their conjectures with logical arguments and proofs. Learning in this mode, according to Collis (1992), leads to 'theoretical knowledge'. This is very different from the mode that is acquired at an earlier age, referred to as Concrete Symbolic. In this concrete symbolic mode, concepts are experienced through the medium of symbolic systems, such as written language, mathematics and musical notation, even though such systems still have ties with the empirical world. Algebraic reasoning in this mode is characterised by a dependence solely on manipulative procedures, such as, simplification of expressions, expansion of brackets, solving simple equations and substituting numerical values for pronumerals (Coady & Pegg 1994). As responses in this mode are linked closely to real world experiences or observations, any processes carried out in this mode may be authenticated with reference to the real world, or what is seen as 'real' by the student (Collis & Biggs 1991). Thus a student responding to a question in a concrete symbolic manner does not recognise any constraints imposed by the mathematical system, nor does the consideration of possible alternatives take place once a conclusion has been reached.

The remainder of this paper explores the extent to which older adolescent students have developed a formal level of reasoning, as described in the SOLO model, and also identifies particular characteristics of this higher-order level of functioning pertaining to the algebraic topics of variable substitution, function notation and algebraic fractions. The following discussion emanates from students' responses to what they perceived as non-traditional (in the text-book sense) questions from the topic areas mentioned above.

Methodology and Sample

A series of questions, related to the three topic areas, in the form of a pencil and paper test, were given to approximately 150 first-year university students, all of whom were enrolled in scientifically-based undergraduate courses. The age group of the sample ranged from 17 - 20 years. Approximately 10% of the sample were interviewed about their responses.

Since it is the intention of this paper to report only on the characteristics identified in those responses classified as formal, the following percentages should be of some interest. They represent the proportion of students' responses within the topic areas to be discussed, that may be classified as formal: variable substitution 45%; function notation 12%; algebraic fractions 46%.

Results and Discussion

A variable substitution question: If $p = 2q$ and $q = st$ find pq in terms of t when $s = \frac{1}{2}$.

Two qualitatively different groups of responses were evident. The first group, classified as non-formal, was confined to making the substitution, $s = \frac{1}{2}$. This was the full extent of the processing of this question.

The second group, classified as formal, was able to utilise variable substitution, albeit with differing levels of sophistication, and it is this ability to "... conceive of an algebraic expression as a process ..." (Tall, citied in Kieran 1992 p. 393) that is indicative of higher-level thought processes. In fact, by using the number of substitutions made as a means of differentiating between response levels, three different levels of responses were able to be identified. Typical examples included:

Level 1:

"$p = 2q$

$= 2st$

$= 2 \times \frac{1}{2}t$

$= t$"

"$q = st$

$= \frac{1}{2}t$

$p = 2q$

$= 2 \times \frac{1}{2}t$

$= t$"

The focus at this level was on a single variable substitution only, after which the processing of the question terminated.

Level 2:

"$pq = 2qst$

$= 2q\ \frac{1}{2}t$

$= qt$"

"$pq = 2qst$

$= 2q^2$"

At this level of response students demonstrated their ability to work with *pq* from the outset, although they later became lost in the symbolism.

Level 3:

"$p = 2q$

$= 2st$

$= 2 \times \frac{1}{2}t$

$= t$

$pq = stt$

$= \frac{1}{2}t^2$"

"$pq = 2qst$

$= 2stst$

$= 2s^2t^2$

$= 2 \times \frac{1}{4}t^2$

$= \frac{1}{2}t^2$"

Clearly this third group of responses indicated the presence of an overall strategy designed to achieve the desired result.

A function notation question: If $f(1) = 5$ and $f(x + 1) = 2f(x)$, find the value of $f(3)$.

Once again a clear dichotomy in the students' responses was observed.

The first group, classified as non-formal, indicated that the students had only a superficial understanding of function notation. Their responses usually involved the substitution of $x = 3$, followed by some rather extraordinary calculations, thus reflecting little understanding of the interrelationships within the system. For example:

$$\text{“}f(3 + 1) = 2f(3)$$

$$\frac{f(4)}{2} = f(3)\text{”}$$

$$\text{“}f(x) = \frac{f(x + 1)}{2}$$

$$f(1) = \frac{f(1 + 1)}{2} = f(1) = 5$$

$$f(3) = \frac{f(3 + 1)}{2} = f(2)\text{”}$$

Furthermore, the students' use of incorrect manipulative procedures was generally characteristic of this type of response.

The second group of responses, classified as formal, indicated that students had a more profound knowledge of the concepts involved in function notation, extending well beyond the simple rearrangement of the formula or the substitution of numbers.

As illustrated below, three levels of response were able to be distinguished.

Level 1:

$$\begin{aligned} \text{“}f(1) &= 5 \\ f(x + 1) &= 2f(x) \\ f(2) &= 2 \times 5 = 10\text{”} \end{aligned}$$

Responses at this level indicated that students could work within the function notation framework but with one aspect only. Hence, answers were brief and to the point, and involved no additional processing.

Level 2: "$f(x + 1) = 2f(x)$

Let $x = 0$

$f(1) = 2f(0)$

$2f(0) = 5$

$f(0) = 2.5$

$f(3) = 7.5$"

"$f(1) = 5$

$f(x + 1) = 2f(x)$

$f(2 + 1) = 2f(2)$

$f(3) = 2f(2)$"

Students responding at this level again were able to work within the structure afforded by function notation although the need to perform many independent steps indicated the lack of any overall strategy. Students appeared to select each step almost at random and while very often these were valid, they achieved no logical purpose in terms of answering the question correctly.

Level 3: "$f(1 + 1) = 2f(1)$

$f(2) = 10$

$f(2 + 1) = 2f(2)$

$f(3) = 20$"

"$f(3) = 2f(2)$

$= 2[2f(1)]$

$= 2[2 \times 5]$

$= 20$"

At this level, the responses indicated that students were capable of applying the concepts underlying function notation as well as identifying and using the interrelationships existing within the question.

An algebraic fraction question: If $0 \leq x \leq 8$, discuss the possible values of x given $\frac{\sqrt{x-4}}{2x-10}$.

This question elicited responses that fell into one of two broad categorisations. The first group was limited to those responses involving either elementary symbol manipulation such as "$\frac{(x-4)^2}{2(x-5)}$", or the substitution of values for x that produced a list of values for $\frac{\sqrt{x-4}}{2x-10}$. The actual number of elements in this list varied according to the number of values of x selected for substitution.

The second group of responses indicated that students had the ability to consider the various limitations (both implicit and explicit) placed on the values of the pronumeral. While this ability marked the onset of formal reasoning, the degree to which these higher-order skills were developed varied from student to student, with three levels again being discernible.

Level 1: “$x \neq 5$” “$x > 4$”

These exemplars reflect the fact that, at this level, the students' focus was directed at either the constraints determined by the numerator or denominator but not both. Unfortunately, written responses of this quality did not indicate whether the explicit constraint on x ($0 \leq x \leq 8$) was taken into account. However in follow-up interviews, when prompted to review these answers in the light of the question, an oral response suggested that this constraint had been considered (although no attempt was made to rewrite these responses in order to incorporate this constraint).

Level 2: “$x > 4, x \neq 5$”

At this level, students tended to list the restrictions on the variable by examining the numerator and then the denominator in turn, but then did not have the ability to integrate the two. In the interview situation, this independence was confirmed.

Level 3: “$4 \leq x \leq 8, x \neq 5$”

A response at this level consisted of a succinct statement that took into account all possibilities.

Conclusion

This paper has sought to confirm some general characteristics of formal reasoning in algebra and furthermore, to specifically identify the key descriptors of such reasoning in relation to some algebraic topics, such as, variable substitution, function notation and algebraic fractions.

With the responses given to the questions in each of the areas explored, those classified as formal depended heavily upon the students' identification and subsequent utilisation of the relationships inherent in the mathematical systems. Embedded within this was the students' ability firstly, to use an algebraic expression as a mathematical object and secondly, to consider conditions and constraints, while using manipulative techniques only as a ‘tool’ to reach a conclusion. These formal responses contrasted markedly with those responses classified as non-formal, where the prominent characteristic was the reliance on numerical substitution and/or incorrect or inappropriate manipulative algebraic procedures. There was strong evidence that students responding non-formally wished either to work only with numbers or simply to operate on the symbols within the provided mathematical system.

In the case of variable substitution, the structural aspects of algebra, that is the ability to operate on algebraic expressions leading to different algebraic expressions (Kieran 1992), are of paramount importance. An important element in distinguishing the levels of formal reasoning was related to the number and type of variable substitutions made. Furthermore, increased levels of complexity in the structure of the given answers were also noted, with the pivotal factor being the ability to monitor successfully all the variable substitutions required.

When working with function notation, students responding formally first, were able to keep within this framework and secondly, with varying degrees of success, were able to maintain the integrity of the function notation fabric. Dominating this type of question was the ability to devise a strategic plan based on the relationships within the question and then to monitor all these relationships while still working towards the correct goal.

Responding formally to questions involving algebraic fractions required students to consider the constraints and limitations placed upon the variable and to have the ability to integrate these to produce a concise mathematical statement. Failure to contemplate such possibilities in any form, led to responses that were classified as non-formal.

The representative sample of problems discussed in this paper differ in the mathematical skills and techniques required to solve them, in the absolute sense. The students' knowledge of and ability to utilise these skills was exemplified by the variability shown in the written responses. However, in seeking to interpret the causes of this variation, some commonalities in the structure of the solution strategies employed were apparent. (Furthermore, these became even more evident in the subsequent interviews.) Firstly, the students needed to be able to *interpret* the question, in order to understand its implications. While this could assumed to be an automatic action for experienced algebra students, this process proved surprisingly difficult for the students included in this sample. In fact, in many cases the students simply could not comprehend the question. Thus, the essential requirement of gaining an overview of the question was lost completely. This, and to some extent the following phase, were the first clear indicators that students had not reached the high levels of cognitive growth required by formal reasoning.

Secondly, students needed to *focus* on the relationships within the question. This phase may have required the rearrangement of the variable relationship/s into a more useful form, or, the determination of the source of any limitations placed on the potential values of the variable. The former was a requirement of the questions on variable substitution and function notation, while the latter was essential for the question on algebraic fractions, if a meaningful solution was to be obtained. Thirdly, the *identification* and associated ramifications of these relationships must be anticipated before the final phase, that of *integration*, takes place. Once the relationships within the mathematical system were recognised, successful integration appeared either to take the form of a concise statement (as with the question on algebraic fractions) or to involve the utilisation of applicable (and correct) manipulative algebraic procedures in order to attain the correct conclusion. It must be stressed here that the transformation procedures used, were not based on real-world connections, and were used as an 'aid' in achieving the end product.

Teachers in the senior-secondary school years seek to realise the potential for formal reasoning in many of their students, but as Collis (1992) pointed out, this reasoning "... does not generalise to all thinking and may not develop in some students at all" (p. 21).Formal reasoning requires the development of high levels of cognitive growth in both the procedural and the structural aspects of algebra. Clearly, this study has shown that not all students entering university undergraduate programs have attained such levels. The next important step is to identify the role manipulative algebraic skills play in enhancing or detracting from formal responses.

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YEARS 2 AND 3 CHILDREN'S CORRECT-RESPONSE MENTAL STRATEGIES FOR ADDITION AND SUBTRACTION WORD PROBLEMS AND ALGORITHMIC EXERCISES

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This paper reports on a longitudinal study in which 104 children were interviewed with 2 and 3 digit addition and subtraction word problems and algorithmic exercises across years 2 and 3. The strategies students used when correctly answering both word problems and algorithmic exercises were identified and compared within each interview and across the 6 interviews. Analysis of the strategies indicated a greater variety being employed for word problems than for algorithmic exercises. Non traditional procedures were dominant for the first 3 interviews, however, a right-to-left strategy similar to the pen-and-paper algorithms became the most popular strategy by interview 5. This was particularly so for algorithmic presentations.

Research suggests that allowing children to construct their own mental computation procedure, "the process of carrying out arithmetic calculations without the aid of external devices" (Sowder, 1988, p.182), should play a major role in the changing curriculum (e.g., Coburn, 1989; McIntosh, 1992). In the past, the focus of primary mathematics computation has been the traditional pen-and-paper algorithm. Now, there is increasing awareness of the role of mental computation as a valid computational method as well as the contribution it makes to mathematical thinking as a whole (e.g., Reys & Bargen, 1991; Sowder, 1990). Research indicates that children actively engaged in the invention of alternative algorithms develop an understanding and appreciation of the number system as well as the flexibility in number calculations (e.g., Kamii, Lewis, & Jones, 1991; Thompson, 1994). As well, children's construction of mental computation procedures is of practical value, because mental computation is the method primarily used to solve everyday mathematics problems in the real world (e.g., Clarke & Kelly, 1989).

A variety of mental strategies for addition and subtraction has been identified in the literature (Beishuizen, 1993; Carraher, Carraher, & Schliemann, 1987; Ginsburg, Posner, & Russell, 1981; Hope, 1987; Madell, 1985; Resnick, 1986). These are listed and described in Table 1, as categorised by Cooper, Heirdsfield, and Irons (in press). They are listed in what, ideally, appears to be increasing order of power and decreasing load on memory.

The *right to left separation* strategy appears to be similar to the traditional addition and subtraction pen-and-paper algorithms. These algorithms are symbolic procedures which follow set patterns of activity: the numbers are written vertically with place values aligned, the place values are separated, and then, with renaming as required, the computation process moves right to left. The *right to left separation* strategy may not involve carrying or decomposing tens. For instance, mentally computing 25+38 by first adding 5+8=13, then adding 20+30=50, and finally combining 50+13=63 is an example of the *right to left separation* strategy. However, this strategy, in all its forms, does have the following common aspects with the pen-and-paper procedures: both numbers are separated into ones and tens; and the operation moves right to left. No other mental computation strategy has these commonalities. In fact the *aggregation* and *wholistic* strategies are in opposition to the pen-and-paper algorithm procedure in not separating all numbers into their place values.

Table 1

Mental strategies for addition and subtraction

Strategy		Example
Counting		28+35: 28, 29, 30,... (count on)
		64-29: 64, 63, 62, ... (count back)
Separation	*right to left*	28+35: 8+5=10+3, 20+30+10=60, 63
		64-29: 60=50+10, 14-9=5, 50-20=30, 35
	left to right	28+35: 20+30=50, 8+5=13, 63
		64-29: 60-20=40, 40=30+10, 14-9=5, 35
Aggregation	*right to left*	28+35:28+5=33, 33+30=63
		64-29: 64-9=55, 55-20=35
	left to right	28+35: 28+30=58, 58+5=63
		64-29: 64-20=44, 44-9=35
Wholistic	*compensation*	28+35: (28+2)+35=30+35=65, 65-2=63
		64-29: 64-(29+1)=64-30=34, 34+1=35
	levelling	28+35: 30+33=63
		64-29: 65-30=35

Because mental strategies have been proposed as an alternative to pen-and-paper algorithms, research has focused on differences between the *right to left separation* strategy and other strategies. Research findings have indicated that: (1) *left to right separation*, *aggregation* and *wholistic* strategies are more accurately used than the *right to left separation* strategy (e.g., Ginsburg, Posner & Russell, 1981; Kamii, 1989); (2) *left to right* strategies produce more sensible answers than *right to left*.(Carraher, Carraher, and Schliemann, 1987); and (3) *left to right separation* , *aggregation* and *wholistic* strategies are preferred for simulated store and word problems while *right to left separation* strategies (pen-and-paper methods) are preferred for algorithmic exercises (Carraher, Carraher, and Schliemann, 1987). Research has also indicated that instruction in pen-and-paper algorithms interferes with the development of natural strategies and effects mental computation performance (e.g., Ginsburg, Posner & Russell, 1981; Heirdsfield, 1995). Of particular interest is the study of Beishuizen (1993). He showed that, in a situation where pen-and-paper algorithms are not taught, *separation* strategies were favoured by low achievers, but *aggregation* strategies were more powerful (in terms of supporting correct mental computation for larger and more complex numbers).

Exploring the effect of pen-and-paper algorithm instruction on mental strategies was a major motivation for an ARC funded longitudinal study of children's mental computation strategies for addition and subtraction was undertaken by the authors. One hundred and four children remain in the study, and are now in year 6. The children were interviewed at the beginning of year 2 and then twice each year. The children were given 2 and 3 digit mental computation tasks in: (1) word problem form (joining addition and separated, missing-addend and comparison subtraction) and algorithmic exercise form (number computation) form. The correct-response strategy use for 2 and 3 digit addition and subtraction word problems (joining addition and separation subtraction only) for the 6 interviews in years 2 and 3 and the beginning of year 4 has been reported by Cooper, Heirdsfield and Irons (in press). This paper reports on the results for 2 and 3 digit addition and subtraction algorithmic exercises (both vertical and horizontal) in the same interviews and compares responses across the two types of representation. Years 2 and 3 were chosen because these are the years in which Queensland schools teach the pen-and-paper algorithm. Cooper, Heirdsfield and Irons (in press) argued that pen-and-paper algorithm instruction appeared to effect students' spontaneous strategies for word problems. The paper also compares observed effect of pen-and-paper instruction on spontaneous strategies for algorithmic exercises with that for story problems.

Method

Subjects. The subjects were 104 children of varying mathematical abilities (one third each of above average, average, and below average) in 6 primary schools (3 state and 3 Catholic) representing a variety of social backgrounds. They participated in the study for 2 years (from the beginning of grade 2 to the beginning of grade 4.

Instruments. The instrument used was Piaget's revised clinical interview technique. The interview questions consisted of 2 and 3 digit addition and subtraction word problems and algorithmic exercises (presented in horizontal and vertical form). Six different question types are being reported here: 2 digit addition, no regroup; 2 digit addition, with regroup; 3 digit addition, with regroup; 2 digit subtraction, no regroup; 2 digit subtraction, with regroup; and 3 digit subtraction, with regroup. Three presentation formats are also being reported here: word problems (joining addition and separation subtraction); vertical algorithm; and horizontal algorithm. The structure for each interview was to give the questions for each question type and presentation form in increasing order of difficulty until the children's responses indicated they were unable to continue. The previous responses (in the given or previous interview) assisted the interviewer find appropriate starting levels.

Procedure. For the purposes of this paper, the students were interviewed in the beginning, middle, and end of year 2, the beginning and end of year 3, and the beginning of year 4. The children were withdrawn from the classroom and interviewed individually in a separate room. The interviews which lasted for a maximum of 30 minutes were videotaped. The word problems were presented visually in the form of pictures and words, and orally as the interviewer verbalised the question. The algorithmic exercises (horizontal and vertical) were presented visually. The questions were revised after interview 3 to take account of children's knowledge growth, to allow the interviewer to probe for the highest level strategies, and yet limit the interview length to 30 minutes. As a consequence, the some question types and presentation forms were not given in some interviews and less students attempted the easier question-types. In particular, after interview 3, the horizontal presentation form was not given and the vertical presentation form was only given for restricted question types in interviews 5 and 6.

The teachers of the 104 children followed the primary Queensland mathematics syllabus. In years 2 and 3, the syllabus focuses on developing basic addition and subtraction facts and teaching the pen-and-paper algorithm for both addition and subtraction. Thus, there was direct teaching of the pen-and-paper algorithms and no teaching of alternative strategies.

Results

Analysis. The videotapes were transcribed into protocols and the performances of the children for each interview and each question were categorised in terms of the strategies in Table 1. To allow analysis across interviews, strategy categories for questions were collapsed into strategy categories for question-types. For the purposes of this paper, the strategies from Table 1 were restructured into the following categories and given codes 1 through 6: (1) *Counting* - predominant use of counting on or back with or without modelling; (2) *Right to left separated place value* - predominant use of this strategy where the numbers are split using place value and computation proceeds right to left; (3) *Left to right separated place value* - predominant use of this strategy where the numbers are split using place value and computation proceeds left to right; (4) *Aggregation* - predominant use of this strategy where one number is kept whole and the other number is separated into place values and computation proceeds right to left or left to right; (5) *Mixed* - mixture of earlier strategies; and (6)*Wholistic* - use of compensation and levelling, with a mixture of earlier strategies. Each child's interview responses for each question-type were then coded by the researcher using the restructured strategies and these were tabulated and analysed.

Student responses. Table 2 below shows children's correct-response strategy use over the interviews. It should be noted that for interview 3, 14 children were not interviewed. For each interview, each question type was analysed for the percentages who attempted the question, correctly attempted the question and correctly used each strategy (1 to 6). The Table displays the results interview by interview, for each question type and each form of presentation.

Table 2

Students' correct-response strategy use

Question type	Presentation form	% attempted all	% attempted correct	% strategy (correct) 1	2	3	4	5	6
Interview 1									
2 dig + no rgp	Word	55.8	36.6	25.9	0	5.8	2.1	1.9	0.9
	Vertical	40.4	19.2	9.6	0.9	6.7	1.9	0	0
	Horizontal	41.3	25.0	14.4	0	4.8	3.8	0	0
2 dig + rgp	Word	8.7	6.8	0	0	1.9	3.8	0	1.0
	Vertical	6.7	5.7	1.0	1.9	2.9	0	0	0
	Horizontal	7.7	3.9	0	0	1.9	2.0	0	0
2 dig - no rgp	Word	42.3	16.3	8.6	0	2.9	2.9	1.9	1.0
	Vertical	25.0	6.7	3.9	0	1.9	1.0	0	0
	Horizontal	20.2	2.9	1.0	0	1.9	0	0	0
2 dig - rgp	Word	5.7	2.9	0	1.0	0	1.9	0	0
	Vertical	7.7	3.9	0	0	1.9	1.9	0	0
	Horizontal	2.9	0	0	0	0	0	0	0
Interview 2									
2 dig + no rgp	Word	76.0	54.9	18.3	13.6	17.3	2.9	1.0	0
	Vertical	55.8	35.6	12.5	6.7	14.4	1.0	1.0	0
	Horizontal	49.0	33.6	14.5	5.7	9.6	2.9	1.0	0
2 dig + rgp	Word	24.0	14.4	0	3.8	7.7	1.9	1.0	0
	Vertical	13.5	12.5	1.0	2.9	7.7	1.0	0	0
	Horizontal	10.6	5.8	1.0	1.0	2.9	0	1.0	0
2 dig - no rgp	Word	57.7	33.6	5.78	13.3	15.5	1.9	1.9	0
	Vertical	34.6	20.2	1.9	3.9	10.5	0	0	0
	Horizontal	38.5	16.4	5.0	4.8	8.7	1.0	0	0
2 dig - rgp	Word	14.4	3.8	0	0	6.8	2.9	0	0
	Vertical	10.6	3.9	0	0	3.8	0	0	0
	Horizontal	6.7	2.9	0	1.0	1.9	0	0	0
Interview 3									
2 dig + no rgp	Word	75.6	57.8	16.7	11.1	23.3	5.6	1.1	0
	Vertical	43.3	33.3	7.8	10.0	15.6	0	0	0
	Horizontal	58.8	43.3	15.6	10.0	17.8	0	0	0
2 dig + rgp	Word	32.2	22.2	1.1	6.7	5.6	6.7	0	0
	Vertical	15.6	13.4	1.1	3.3	6.7	2.2	0	0
	Horizontal	24.4	13.3	1.1	4.4	4.4	2.2	0	0
2 dig - no rgp	Word	71.1	41.1	7.8	6.7	15.6	1.1	8.9	1.1
	Vertical	26.7	12.2	4.4	7.8	4.4	0	0	0
	Horizontal	44.4	23.3	4.4	5.6	13.3	0	0	0
2 dig - rgp	Word	21.1	6.7	1.1	1.1	2.2	1.1	1.1	0
	Vertical	10.0	4.4	0	0	2.2	2.2	0	0
	Horizontal	15.6	4.5	0	0	0	3.3	0	1.1
Interview 4									
2 dig + no rgp	Word	91.3	83.6	5.7	26.0	44.2	0	0.6	0
2 dig + rgp	Word	71.2	55.8	1.0	18.3	17.3	3.9	14.4	1.0

3 dig + rgp	Word	17.3	12.5	0	0	2.9	1.0	2.9	7.7
2 dig - no rgp	Word	81.7	6.6	6.7	21.1	34.7	1.9	2.9	7.7
2 dig - rgp	Word	50.9	17.9	0	1.7	5.7	5.7	3.8	0.9
3 dig - rgp	Word	2.8	2.8	0	0	0.9	0	0	1.9
Interview 5									
2 dig + rgp	Word	83.7	60.6	0	26.0	19.2	2.3	11.5	1.0
	Vertical	58.7	52.2	0	36.6	2.9	1.0	0	0
3 dig + rgp	Word	36.5	25.0	0	6.7	3.8	1.0	2.9	10.6
	Vertical	44.2	31.7	0	26.9	4.80	0	0	0
2 dig - rgp	Word	72.1	23.1	1.0	9.6	2.9	4.8	3.9	1.0
	Vertical	65.4	18.2	0	16.3	1.9	0	0	0
3 dig - rgp	Word	17.3	11.5	0	1.0	1.9	3.9	1.0	1.2
	Vertical	27.9	12.5	0	8.7	2.9	1.0	0	0
Interview 6									
2 dig + rgp	Word	90.4	72.1	1.0	38.5	9.6	1.9	21.1	0
	Vertical	28.8	17.3	0	17.3	0	0	0	0
3 dig + rgp	Word	69.2	42.3	0	29.8	3.9	0	1.9	6.7
	Vertical	80.8	73.1	0	64.3	5.7	0	2.9	0
2 dig - rgp	Word	79.8	42.3	0	21.2	3.9	3.8	7.7	5.8
	Vertical	62.5	28.9	0	26.0	1.0	1.0	1.0	0
3 dig - rgp	Word	28.8	14.4	0	5.8	1.9	1.0	1.9	3.8
	Vertical	51.9	30.8	0	25.9	2.9	0	1.9	0

Discussion

Over the two years, an increasing percentage of students attempted the questions, and accuracy levels improved. A variety of strategies was identified over the interviews, particularly for word problems (similar to Carraher, Carraher & Schliemann, 1987). As well, addition was attempted with higher frequency and accuracy than subtraction, indicating the difficulty students have with subtraction.

Word problem responses. The percentage of children attempting more difficult questions and the percentage of children who were correct in more difficult questions increased over the interviews. The percentage of children correctly using *counting* decreased over the interviews, while the percentage using *separated, aggregation* and *wholistic* strategies increased. The students attempting harder examples earlier tended to use higher numbered strategies (for instance, *left to right, aggregation*, and *wholistic*) and be more accurate with them. The *separated* strategies were more popular than *aggregation*, although they resulted in less accuracy (a similar result to Beishuizen, 1993 - explanations could be that the *aggregation* strategies require less effort from working memory or only the better students used these strategies). As is described in Cooper, Heirdsfield, and Irons (in press), *left to right separated place value* initially replaced *count* as the most popular correctly used strategy; whereas, by interview 6, *right to left separated place value* was the dominant strategy and had generally a higher level of accuracy.

Algorithmic exercise responses. Children's performance on vertical addition algorithmic exercises improved dramatically over the two years. Increases in performance were as marked for subtraction but at a lower level. Overall, *counting* was the predominant strategy in the first two interviews, although it was rarely used for complex examples, which were solved using more powerful strategies. *Left to right separated place value* became the dominant strategy in interview 3 for all question types but was replaced by the *right to left separated place value* strategy in the last two interviews. This shift was very marked as was the decrease in the use of the *aggregation*, *mixed* and *wholistic* strategies.

There appeared to be no discernible pattern of difference between children's use of *right to left separated place value* and *left to right separated place value* strategies between vertical and horizontal presentations. Surprisingly, *right to left separated place value* was sometimes used more for the horizontal than the vertical.

Comparing responses between presentations. Generally word problems were attempted by a higher percentage of students than algorithmic exercises and resulted in more success, except for the vertical algorithm in interview 6 where the *right to left separated place value* strategy is dominant. Word problems elicited the greater variety of strategies. The move to *right to left separated place value* and the decline in the use of other strategies was significantly more pronounced for algorithmic exercises than for word problems.

Instructional effects. As described earlier, the pen-and-paper procedure for addition and subtraction algorithms has common aspects with the *right to left separated place value* strategy for mental addition and subtraction and no other mental computation strategy has these commonalities. The results from the interviews show that, in year 3, there is strong increase in popularity for the *right to left separated place value* strategy (as Madell, 1985, also found) and that this is most pronounced (and most successful) for the vertical algorithm presentation form. This focus on right to left was not as marked in Beishuizen's (1993) findings where the pen-and-paper algorithms are not taught. Students appeared to be very familiar with the *right to left separated place value* strategy in interview 6 (one explanation of its success could be that this familiarity requires less effort from working memory). Therefore, the children's responses to the word problems and the algorithmic exercises, and the relationship between these responses, is a strong indication that there is an instructional effect. That is, instruction in pen-and-paper procedures influences children to choose the *right to left separated place value* strategy for mental addition and subtraction.

Conclusions

In summary, the research reported in this paper indicates that children can efficiently and effectively compute with their own non traditional mental algorithms before instruction in pen-and-paper algorithmic procedures , and many children use a variety of approaches. Further, the pen-and-paper related strategy (*right to left separated place value*) became a dominant strategy only after instruction. That is, this strategy is not a typical one for children to employ, rather, left to right procedures are more natural. As well, the practice of teaching algorithmic presentations before word problems does not reflect children's inherent capabilities. It appears imperative that children be given the chance to build on their own natural skills by choosing and verbalising mental strategies. Less emphasis should be placed on the traditional pen-and-paper algorithm, and more emphasis on developing children's legitimate spontaneous strategies in a problem solving environment.

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Why the Psychological must Consider the Social in Promoting Equity and Social Justice in Mathematics Education

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Abstract

This paper suggests that the recent concentration of psychological studies in mathematics education on the individual at the expense of the social cannot support the development of equity and social justice in and outside mathematics classrooms. We argue that an alternative research agenda is necessary, an agenda which sees the social as paramount. Such an agenda would build a theoretical base for what has been termed social constructivism. We aim to begin the process of setting such an alternative agenda.

Setting the agenda

> *Mathematics is the one subject that most of us can imagine as far removed from issues of politics, value, professional ethics and democracy.*
>
> *(Applebaum, 1995, p2)*

This is a view we hear reflected in many of the classrooms in which we work with teachers and learners of mathematics, indeed it seems to be a view held by many teacher educators both outside and inside the subject discipline of mathematics. We see the creation of the National Curriculum within the UK as the culmination of a process in which educators and others who hold control over the curriculum are attempting to find the most appropriate order in which to teach mathematical skills. The idea that there exists a hierarchical order in which mathematical skills can be taught being taken as given. In this way the purpose of teaching becomes the delivery of this prescribed curriculum and the function of research is to make this process as smooth as possible. This functional view of research into mathematics education in turn moves school mathematics further and further from any social domain. Applebaum criticises this approach to research, stating

> *Because mathematics education research has focused primarily on classroom activities, optimal sequence of knowledge topics, and the cognitive development of individuals, it has implicitly constructed a stark distinction between school mathematics and the world outside of*

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> *schools, placing popular and mass culture, the public sphere, and related issues of power and politics in the background.*
>
> *(Applebaum 1995, p52)*

The aim of this paper is to set an agenda for research in mathematics education which takes as given the fact that learning mathematics in schools is a social, a cultural and a political act. Far from being value free, apolitical and removed from the social world which we inhabit, mathematics has a crucial role to play in the way in which our societies and democracies develop and the roles which the learners in our care will follow in these societies. Jerome Bruner suggested that all classrooms contain and transmit values held by the teachers in those classrooms.

> *I do not for a minute believe that one can teach even mathematics or physics without transmitting a sense of stance toward nature and toward the use of mind.*
>
> *(Bruner, 1986, p128)*

He argues that teachers operate very differently when they are working in classrooms than in other social situations presenting a "far less hypothetical, far less negociatory world than the one they offered to their colleagues. (op cit, p126)

Jenny Maxwell suggests that "mathematics (in school) lends itself to an authoritarian teaching approach which fosters mystique, creating fear and panic" and furthermore this is readily accepted by many learners (Maxwell, 1989, p 222). Why is it that so many learners take for granted a transmitional pedagogy for the learning of mathematics, even though many see themselves as mathematical failures? To some extent many within our communities take on a fatalistic approach to their (non)development as mathematicians. Studies of teacher and pupil attitudes towards their own mathematical learning and the ideologies underpinning individual ideas of the most appropriate pedagogical approaches could offer a way to undermine this taken-for granted view of mathematics learning and teaching.

There is evidence that teachers will continue with practices which have failed them as learners of mathematics, indeed Joe Relich (Relich, 1992) at PME XVI presented a paper which suggested that teachers of primary age children can suffer from a low self-concept of themselves as mathematicians which in turn effects the learning of mathematics within their classrooms. In this case the culture of school mathematics can be seen to be recreating itself. As Bruner suggests,

> *The language of education is the language of culture creating, not of knowledge consuming or knowledge acquisition alone.*
>
> *(Bruner, 1986, p133)*

The world of school mathematics becomes an accepted culture by learners within these classrooms. They adopt the values of the mathematics curriculum and can be resistant to any challenge to these values. We have numerous examples of novice secondary mathematics teachers speaking in some frustration about how their pupils resist all attempts to wean them away from their individualised learning scheme. This resistance to change is reported by Sue Willis when she describes the reaction of some learners to an innovative course offered by a teacher in a school she was working in

> *It was predominantly the high achieving male students who were most reluctant to participate in the group work that was an essential component of the course. These observations demonstrate how, unless we are vigilant, those already privileged in mathematics can continue to control and define what constitutes mathematics in school.*
>
> *(Willis 1995, p195)*

Hence the norms regulating accepted practices in the classroom depend not just on a teacher's chosen teaching strategies, but on the nature of social power and the implicit control this allows the powerful to exert. It may not always be the teacher who holds the power; power can be exercised by certain privileged groups, e.g. males, the middle class, school management, the media and so on. Another key area of research in our view would examine how the power-less can become more power-ful through the learning and teaching of mathematics.

Emancipation

In a recent PME paper, Barbara Jaworski considered the extent to which engaging in "critical reflection" supported the emancipation of teachers

> *Teacher emancipation, [...] arises consciously from teachers becoming aware of their own knowledge and purpose through critical enquiry into their practice. Emancipation seems to be a state within the liberating process of action oriented enquiry into their practice.*
>
> *(Jaworski, 1992 p 289)*

For us, two terms here need further elaboration: *critical* and *emancipation*. Jaworski uses the term 'critical reflection' following Van Manen (1977) to refer to the ethical and moral dimension of teaching. In this vein she makes the case

> *that reflective practice in mathematics teaching, which is critical and demands action, is a liberating force, and that teachers engaging in such reflection are emancipated practitioners*
>
> *(Jaworski 1992, p 290)*

We want to use the word 'critical' in a wider sense to involve the willingness to transcend taken-for-granted assumptions as well as to look at the power relations existing within the educative context. Jaworski mentions Smyth who similarly advocates a "critical pedagogy of schooling which goes considerably beyond a reflective approach to teaching" (Smyth 1985).

Jaworski further elaborates her use of the word emancipation in describing part of the problem as

> *teachers who are bound by tradition, convention or curriculum and who fail to perceive their own power to tackle constraints. The result for pupils is likely to be a limited or impoverished curriculum*
>
> *(Jaworski, 1992, p289)*

Again we would suggest a notion of emancipation which has considerably wider implications. The emancipation of the teacher and pupils described by Jaworski can be seen as a form of local control over classroom decision making. Restricting attention to such 'local' decision making without a recognition of how power is exerted in society may result in an inability to challenge the 'taken-for-granted' assumptions. Teachers are after all working within an educative context in which both they and their pupils are excluded from most decisions which affect them: decisions about working conditions, class size, pay, power relations, pupil discrimination and so on. A broader view of emancipation would echo Patti Lather's call for emancipatory research which 'allows us to understand the maldistribution of power and resources underlying our society' and also is a part of the process of the development of 'a more equal world'. (Lather, 1986, p258)

In a discussion of the treatment of pupils of Mexican descent, Khisty cited research showing that pre-service teachers made decisions on hypothetical pupils based solely on the social class and ethnicity of pupils (Khisty 1994, p 91 citing Licón 1979)). Whilst this was based on pre-service teachers in the USA, the prevalence of so many working class pupils in so called 'remedial' classes in the UK is suggestive of a wider phenomenon. Cecile Wright's work can be seen as supporting the view that many Black pupils are not placed in mathematics sets purely on what some may describe as 'mathematical ability.' (Wright, 1986, 1994)

Khisty goes on to argue that promoting equity in mathematics education requires a move beyond content and psychological factors to acknowledge social psychological factors also. Promoting equity requires a recognition that mathematics is not value free, culturally neutral or politically impartial. We go further and consider that mathematics *teaching* is a political act. It involves wielding power and authority; it is about how relationships with and between pupils will be established.

Politics and ideology

At least in the U.K. it would appear to us that mathematics teachers rarely see themselves as taking part in a political process, a perspective which falls to challenge the status quo; fails to challenge the powerful. Paul Ernest has written of how certain classroom practices and beliefs are consistent with particular ideological positions (Ernest 1991). To say that mathematics teaching is not political is itself a political statement. That 'no ideology' is an ideology is seen very clearly in the calls from politicians of all persuasions to concentrate on what they term the basics of education, as Henri Giroux points out:

> *This discourse ("back-to-basics") invokes forms of institutional authority that says little about issues of equity or social justice; it is the view of authority rooted in an unproblematic appeal to the rules and to the imperatives of individual successes*
>
> *(Giroux, 1989, p19)*

(See also Claudia Zaslavsky 1981) This individualisation of the curriculum is particularly seen in mathematics classrooms, the rapid growth of individualised learning schemes can in itself be seen as an ideology and the following passages from Michael Apple may have particular resonance to mathematics classrooms with their view of teacher as problem solver, and the common-sense belief within many schools in the UK that setting is the only way to teach mathematics effectively.

> *one can examine schools and our action on them in two ways: first, as a form of amelioration and problem-solving by which we assist individual students to get ahead; and second, on a much larger scale, to see the patterns of the kinds of individuals who get ahead and the latent outcomes of the institutions. These larger social patterns and outcomes may tell us much about how the school functions in reproduction, a function that may tend to be all too hidden if our individual acts of helping remain our primary focus.* *(Apple, 1982,p13)*

> *The curricular and pedagogic practices that are used to organise the routines in most schools - the differentiated curriculum, the grouping practices, the hidden curriculum - do play a part in enabling students to internalise failure based on this sorting process as an individual problem. (It is my fault. If only I had tried harder).*
>
> *(Apple, op cit., p59)*

So how can mathematics educators and Mathematics education researchers move to create an alternative culture within our classrooms which acknowledges the political nature of the classrooms in which we work? We would support Nel Noddings in her call for the development of an ethical framework which would underpin constructivism as a theoretical standpoint and in turn support moves towards the politicisation of the mathematics curriculum.

> *... constructivism as a pedagogical orientation has to be embedded in an ethical or political framework. The primary aim of mathematics teachers cannot be to promote mathematical growth, although that is*

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> *certainly one worthy goal. Rather, the primary aim of every teacher must be to promote the growth of students as competent, caring, loving and loveable people. Teachers with this aim will work flexibly in teaching mathematics - inspiring those who care about mathematics for itself to inquire ever more deeply, helping those who care instrumentally about mathematics to prepare for the line of work they desire, and supporting as best they can those students who wish they never had to encounter mathematics. To have uniformly high expectations for all students in mathematics is morally wrong and pedagogically disastrous. It is part of a sloganised attempt to make our schools look democratic and egalitarian, when in fact, they are systems continually struggling for tighter control.*
>
> *(Noddings 1993, p)*

We would suggest that a research agenda driven by the aims described here would look radically different from our present agenda. However, a view of constructivism which does not take into account the social, and sees instead its focus as the lone individual making sense of the world could perhaps inadvertently serve to eradicate any threat to the political status quo by fostering cults of the individual. One manifestation of this is the present 'construction' of *radical constructivism*:

> *Although a number of different forms of constructivism exist, the radical version most strongly prioritises the individual aspects of learning. It thus regards other aspects such as the social, to be merely a part of, or reducible to, the individual.*
>
> *(Ernest 1994, p304)*

In this PME paper, Paul Ernest considers the nature of social constructivism. He argues, along with others, for a theory of learning which rejects Piagetian roots and looks towards Vygotskian theory of development. Whilst this may be seen as a step forward, it is still locked into the intra-individual:

> *Vygotskian versions of social constructivism suggest (...) the import of the overall social context of the mathematics classroom as a complex, organised form of life including (a) persons, relationships and roles, (b) material resources, (c) the discourse of school mathematics, both content and modes of communication*
>
> *(op cit, p31)*

A further manifestation of this cult of the individual is the great attention being given to encouraging reflection, where an individual is encouraged to look at their own practice and find ways in which they can make changes without any consideration of why things should be as they are in the world.

Erna Yackel raises the need for a wider perspective in her PME paper (Yackel 1994) when she discusses the implications of a research and innovation project in two culturally different settings (a "rural/suburban white middle class comunity" and an "inner city minority community"). She warns that as researchers we may ignore the effects of the culturally-situated communities and bring about "even greater disparities in the type of mathematics education children experience" (p 391). We agree with her call to "clarify the ideological, social and political dimensions of our efforts to initiate reform in mathematics education".

Conclusion

We are arguing here that a separation of the psychological and the social presents an unhelpful way of setting an agenda for a mathematics education for a fair and just society. We would hope that the questions we raise in this paper can be a part of an impetus towards social research in mathematics education. One of the main functions of education is social regeneration; it is therefore linked to politics and power - keeping the powerful powerful and the powerless powerless. The struggle for a more just and equitable society needs us to explore the political dimensions of mathematics teaching and learning and through these social dimensions arrive at the very heart of teaching and learning theories.

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CLINICAL ANALYSIS OF DIFFICULTIES IN ALGEBRAIC PROBLEM SOLVING AMONG BRAZILIAN STUDENTS: PRINCIPAL ASPECTS AND DIDACTIC ISSUES

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This research is aimed to complete a previous set of descriptive data concerning categories of difficulties in elementary algebra among Brazilian students with a clinical analysis of two main sets of difficulties, related to algebraic problem-solving and equation processing, respectively. This analysis suggests difficulties in problem representation (passage from natural language to formal representation), but also in the comprehension of the principles involved in algebraic processing. Protocol analysis is provided to illustrate these difficulties, and some didactic issues are discussed towards the proposition of an effective didactic sequence in algebra.

In a previous study, Da Rocha Falcão (1995a) analyzed the protocols of 459 thirteen to seventeen year-old Brazilian students from three states of Brazilian federation, submitted to a series of 20 problems in elementary algebra, covering 10 algebraic structures, each of these structures (except structure 5 - see Table 1 in the next page) presented as a word-problem and an equation to be solved. This first study showed a consistent pattern of difficulties, only described in terms of statistical frequencies of categories. In the present paper, 22 more students were added to the original sample and a clinical analysis of the difficulties mentioned above is provided, in order to offer further information about the cognitive task involved in reaching the conceptual field of algebra.

Category data analysis of the set of 481 protocols shows four main sources of difficulties in word-problem questions, and three sources in equations, as summarized in Table 1. The categories of errors appearing in the referred table will be commented and exemplified in the next sections. Nevertheless, two aspects must be immediately mentioned, the first of them being somewhat deviant in comparison with other studies (e.g., Kieran, 1989) : 1. there is no statistically significant difference between the level of difficulty (average percentage of errors for algebraic structure) of word-problems and equations (Wilcoxon Matched-Pairs Signed Ranks Test, 2-tailed p = .3743). Our present explicative hypothesis is that this result is probably due to the fact that Brazilian students, like others, have important difficulties in dealing with algebraic word-problems, but seem to have more specific difficulties in algebraic processing than students from other countries. 2. Arithmetic processing is an important source of errors in algebraic problem-solving (this aspect had already been mentioned in other studies, e.g., Kieran, 1991, Herscovics & Linchevski, 1991, Chaiklin & Lesgold,1984)

Data displayed in Table 1 shows that modeling and algebraic processing were the two most important sources of difficulties in word-problems and equations, respectively, and will be clinically described and analyzed in the following sections.

1. Modeling: this source of difficulty referred to the proposition of a previous explicit representation (graphic scheme, list of topics in natural language or equation) to a word-problem given. Three levels of difficulties were observed in the context of modeling difficulties:

Algebraic structures	Categories of errors (%)								
	Problems					**Equations**			
		Categories of errors					Categories of errors		
↓	% Diff.	M1	M2	Ag	Ar	% Diff.	Ag1	Ag2	Ar
1. $x + a = b$	**6,2**	17	0	77	6	**17,5**	31	64	5
2. $ax + b = c$	**39,6**	41	16	7	36	**19,3**	50	42	8
3. $ax + b = cx + d$	**68,2**	65	8	2	25	**24,7**	50	48	2
4 $ax + b + y = cx + d + y$	**71,1**	70	11	4	15	**51,1**	70	30	0
5. $a(bx+c)+d = e(fx+g)+h$	(No corresponding problem)					**40,3**	42	54	4
6. $x/a = b$	**24,5**	51	14	11	24	**43,0**	60	37	3
7. $a/x = b$	**22,9**	42	17	33	8	**39,1**	60	36	4
8. $(x + a)/b = c$	**52,4**	40	16	10	34	**47,7**	68	27	5
9. $x + y = a$ $x - y = b$	**69,8**	37	13	18	32	**69,2**	60	36	4
10. $ax + by + c = d$ $ex + fy + g = h$	**86,1**	76	9	7	8	**76,3**	70	30	0

Table 1

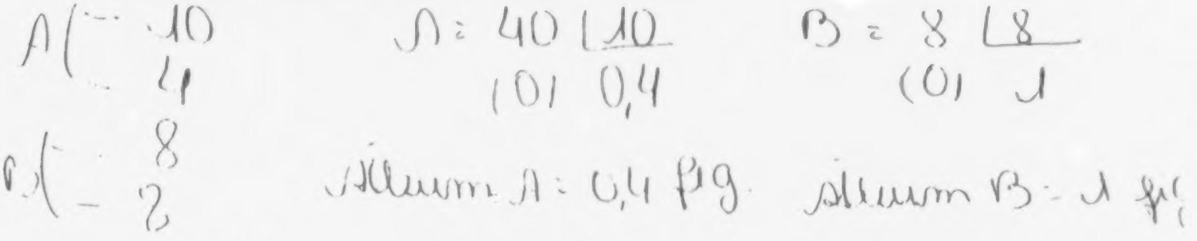
Percentual distribution of categories of errors in word-problems and equations among thirteen to seventeen year-old Brazilian students (n = 481).

LEGEND: % Diff. = Average percentage of errors for a given structure, **M1** = no formal (equation) previous representation proposed, **M2** = equation proposed, incorrect modeling of the problem, **Ag** = error due to algebraic processing, **Ar** = error due to arithmetic processing, **Ag1** = Interruption of algebraic processing without any answer, **Ag2** = error due to algebraic processing. **x** , **y** = unknown quantities, **a, b, c, d, e, f, g, h** = known numbers

1.1. M1: Schematic, pre-equation representation of the problem, with incorrect comprehension of the mathematical relations between known and unknown entities referred by the problem's enunciate, as exemplified by the transcription below, from M.P. (16 years-old, 1st. grade high-school):

Problem proposed: *Vera and Joana have the same number of self-adhering stickers. Vera has 10 albums of stickers and 4 more stickers, while Joana has 8 similar albums and 8 more stickers. How many stickers are there in each album?*

M.P. representation and answering of the problem:



The answer proposed by M.P. is *0,4 sticker* in album A, and *1 sticker* in album B. As noticed in many other protocols, the semantic anomaly of "0,4 *sticker*" is not taken into account as a sort of metacognitif alarm. She starts representing albums A and B, with the number of albums and extra stamps for each album, according to the enunciate of the problem. She decides, then, to divide the number of extra stamps by the number of albums for both albums (A and B), as if the number of extra stamps represented the total of stamps.

1.2. M2: Proposition of an inadequate equation to the problem, as exemplified by the transcription below, extracted from the protocol of C.S.S. (15 years-old, 1st grade high-school):

Problem proposed: the same previously mentioned (Vera and Joana's)

C.S.S. representation and answering of the problem:

$$\begin{cases} x = 10 + 4 \rightarrow x = 14 \\ y = 8 + 8 \rightarrow y = 16 \end{cases}$$

R: No album de Vera havia 14 figurinhas
No album de Joana havia 16 figurinhas

• TRANSLATION *There were 14 stickers in Vera's Album*
There were 16 stickers in Joana's Album

C.S.S. proposes a system of equations in order to represent the problem. The unknowns x and y, in this system, are intended to represent the number of stickers in albums A and B, respectively. The fact that x = y, according to the problem, is not represented by C.C.S., neither the distinction between sets of stickers (albums) and individual stickers, what makes her sum numbers of albums and individual stickers (10 + 4 and 8 + 8)

2. Algebraic processing: this topic refers to the work of equation transformation following a certain set of syntactic, formal. Both problems and equation tasks showed important difficulties in this domain, as exemplified in the following sections and sub-sections:

2.1. Ag, Ag1 and Ag2: Error due to algebraic processing after the proposal of an adequate equation for a given word-problem (Ag), or after having received an equation to solve. Three important sub-sets of difficulties appeared here:

2.1.1. Algebraic-arithmetic processing:

Algebraic structure 5: $8(2x + 4) + 3 = 4(5x + 2) + 7$

C.C. protocol (16 years-old, 1st grade high-school):

$$20x + 3 = 22x + 7$$

$$20x - 22x = 7 - 3$$

$$2x = 4 \quad x = \frac{4}{2} \quad \boxed{x = 2}$$

Comment: C.C. reduces 8(2x + 4) to 20x: he first multiplies 8 by 2x, having 16x, which is directly summed with 4; he repeats consistently this procedure

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to the other expression [4(5x + 2)].

O.V.G. protocol (16 years-old, 1st grade high-school):

$8x$
$-4x$
$4x$

7. $8(2x + 4) + 3 = 4(5x + 2) + 7$

$16 + 8x + 32 + 3 = 20 + 4x + 8 + 7$

$\underline{16} + 8x + \underline{32} + 3 - 20 - 4x - 8 - 7 = 0$

$4x + 51 - 35 = 0$

$4x = 16$

$x = \frac{16}{4}$

$\{x = 4\}$ resultado $(x = 4)$

$+32$
$+16$
$+3$
51

Comment: O.V.G. operates separately over the numeric and literal parts of 2x and 5x, what explains the decomposition of 8(2x + 4) into 16 + 8x + 32 in the first line of his script.

Algebraic structure 8: $\frac{x + 12}{8} = 10$

S.J. protocol (15 years-old, 1st grade high-school): tenham 17 camisas

$\frac{x + 12}{8} = 10$

$\frac{x + 3}{2} = 10$

$x + 3 = 10 \times 2$

$x + 3 = 20$

$x = 20 - 3$

$x = 17$

Comment: S.J. decides to *simplify* the fraction in the left term of the equation, dividing the denominator and the numerical part of the numerator by 4.

The three examples of protocols above illustrates the coexistence of algebraic (the comprehension of entities involving numbers and unknowns, like 5x, and their operational coexistence with numbers) and arithmetic difficulties (operational priorities, distributive property of addition in relation with multiplication).

2.1.2. Comprehension of the equivalence principle (and the algebraic meaning of the equal sign) in equation processing:

Algebraic structure 8: $\frac{x + 12}{8} = 10$

E.B.L. protocol (17 years-old, 1st grade high-school):

$8x + 96 = 10$

$8x = -96 + 10$

$8x = -76$

$x = \frac{-76}{8} = \boxed{-9,5}$

Comment: Like S.J., in the previous example, E.B.L. decides to simplify the left term of the equation, but he does it differently, multiplying (incorrectly) this term by 8; this action is restricted to the left term, as if the multiplication was a "local" initiative, unnecessary for the right term of the equation. The equivalence principle is once more violated in the next transformations, probably guided by the "move from a side to the other" principle.

A particular use of this principle seems to explain the error exemplified by the transcription on the right, from the protocol of A.R. (no age record, 1st grade high-school), solving an equation corresponding to the algebraic structure 6: $\frac{x}{10} = 8$

$$\frac{x}{10} = 8$$

$$x = \frac{8}{10} = \frac{4}{5}$$

The failure in building up the comprehension of the principle of equivalence explains also the production below:

Algebraic structure 4: $10x + 4 + y = 8x + 8 + y$

P.C.S protocol (no age record, 1st grade high-school):

$$10x + 4 + y = 8x + 8 + y$$

$$10x + 8x = 4 + 8 - y + y$$

$$18x = 12 = 2y$$

$$18x + 12 = 2y$$

$$30x = 2y$$

Comment: The use of the equal sign in P.C.S. work shows clearly the difficulty in considering an equation as a mathematical function, represented by the equal sign; she proposes, then, an expression with three terms (and two equal signs). Furthermore, she sums 18x and 12, repeating a frequent pattern of error already mentioned.

$$10x + 4 = 8x + 8$$

$$14x = 16x$$

$$x + x = 14 - 16$$

$$2x = -2$$

$$x = \frac{-2}{2} = \boxed{x = -1}$$

This addition of numbers and number-literal terms is recurrent, as shown in the reproduction on the left, from the same protocol (algebraic structure 3: $10x + 4 = 8x + 8$).

2.1.3. Comprehension of both substitution and addition algorithms for the processing of systems of equations

Algebraic structure 9: $x + y = 38$
$x - y = 6$

J.C.C.M protocol (16 years-old, 1st grade high-school):

$$\begin{cases} x + y = 38 & x \\ x - y = 6 & x = 6 + y \end{cases}$$

$$6 + y - y = 6 \qquad 0 + y = 38$$

$$-y + y = 6 - 6 \qquad y = 38$$

$$x = 0$$

$$S = (0, 38)$$

Comment: J.C.C.M substitutes the value of x extracted from the second equation into the same equation, what leads him to the identity 0 = 0, changed by x = 0.

E.C.S.B. protocol (16 years-old, 1st grade high-school):

$$\begin{cases} x + y = 38 \\ x - y = 6 \end{cases}$$

Comment: This girl processes the system of equation as if it was a proportion structure a:b::c:d, making then appeal to the property of the "crossed multiplication" (ad = cb)

$6 \cdot (x + y)$ [illegible]

$6x + 6y \quad 38x \quad 38y$

$6x \quad 38x \quad 38y \quad 6y$

$32x \quad 44y \quad (-1)$

$|32x = 44y|$

R.B.C.S. protocol (15 years-old, 1st grade high-school):

$$\begin{cases} x + y = 38 \quad (\times 1) \\ x - y = 6 \end{cases}$$

$$\begin{cases} -\cancel{x} - y = 38 \\ \cancel{x} - y = \end{cases}$$

$-2y = 32$

$-\alpha \quad y = \frac{32}{2} = \boxed{16}$

logo: $x + 16 = 6$

$x = 6 + 16$

$x = \boxed{22}$

$\begin{array}{r} 16 \\ +6 \\ \hline 22 \end{array}$

Comment: This student combine difficulties related to the comprehension and utilization of the principle of equivalence, with difficulties directly related to the utilization of the addition algorithm for solving a system of equations. Her first initiative, consisting in multiplying the first equation by -1, is restricted to the literal portion of this equation. The "disappearance" of the minus sign in the passage from the expression $-2y = 32$ to the expression $y = \frac{32}{2} = 16$ is perhaps explained, once more, by a partial comprehension of the "*change side, change sign*" principle.

• Conclusions and didactic issues

The clinical analysis briefly resumed in this paper suggests important difficulties in the comprehension and use of algebra as a *representational tool* in the particular sample studied, which is coherent with data from other countries, e.g. Mexico (Gallardo & Rojano, 1988). Nevertheless, differently of data from these countries, Brazilian data suggests also very important difficulties in algebra as a formal, syntactic principle-governed system: these students show difficulties in generating a suitable equation in order to solve an algebraic word-problem, but they have also important difficulties in processing an equation proposed in the context of class-activity in mathematics. As we have briefly shown, clinical analysis of protocols suggests a prevailing use of algebra as a syntactic, abstract rule-governed system, with large prevalence of the "*change side, change sign*" principle, in spite of the equation equivalence principle. The consequences of this pedagogical option is illustrated by the present data, and reinforces a remark made by Kieran (1992, p.400), for whom "(...) many students who use transposing are not operating on the equation as a mathematical object but rather are blindly applying the change-side - change sign rule". This last aspect is still aggravated by remaining arithmetic problems, in interaction with properly speaking algebraic obstacles. As a last feature in this dark picture, we should mention the permanence of difficulties along school levels up to the 1st degree of high school, when the students are supposed to have understood the fundamentals of algebra and to start using it as a *tool* (Margolinas, 1991; Douady, 1986).

This pedagogical *state of affairs* suggests specific initiatives concerning didactic of elementary algebra. If arithmetic procedure implies an immediate search for solution, represented by the calculation of intermediate values in order to reach a final answer, algebraic procedure, differently, postpone the very activity of solution's search and begins by a formal transposition from empirical domain or natural language to an specific representational system (Da Rocha Falcão, 1995b, 1993). In an introductory or remedial didactic sequence aimed to teach elementary algebra, both representational (modeling) and syntactical (algorithmic) aspects must be simultaneously attacked. In this sense, four main aspects should guide an effort of didactic engineering, aimed to propose a didactic sequence in elementary algebra: 1. Use of culturally-significant metaphors (e.g., two-pan balance scales) as cognitive scaffoldings in order to help making sense of the equivalence principle in problem representation and equation processing, as suggested in Da Rocha Falcão, 1995b; 2. Negotiate and make explicit a new *didactic contract* (Schubauer-Leoni, 1986) for mathematics problem-solving, represented by the principle *represent first, solve latter*. 3. Not to avoid reviewing arithmetic in the context of algebra, in order, for example, to help students move from the conception of a numeric-literal expression as a series of operations to perform, to the formal-algebraic conception of a *polynomial* (Chevallard, 1990). 4. Use of auxiliary tools (e.g., spreadsheets) for making sense of the functional ideas of variable and parameter, and so doing, make it possible for the students to look at an algebraic expression as a ***template***, "(...) a potential arithmetic relationship waiting to be realized" (D.O. Tall, quoted by Ainley, 1995). The operational proposition and didactic evaluation of such an integrated didactic sequence is our main research goal for the two next years, motivated by the important difficulties brought to light by the present study.

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What is the difference between remembering someone posting a letter and remembering the square root of 2?

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I claim that what distinguishes the mathematical thought of people capable of easy mathematical attainment is a heightened use of episodic memory about mathematical objects. To the contrary, people who find mathematics hard and sterile probably do so because their memory of it is semantic in the most limited sense.

Learning mathematics ought to be easy, but it's not. Why is that?

In this paper I want to explore the possibility that, as well as the constrictions of rigour, one reason mathematics is so hard for so many people is that mathematicians do not think about mathematics the way that other people do. In order to make this a non-trivial statment, let me re-phrase it differently: there are ways of thinking about mathematics that make it far easier than other ways, and most people adopt the harder way. Moreover, by the very nature of their habits of thought - their habitual way of constructing their world - they are highly unlikely to ever see that there is an easier way. The idea that some people view mathematics in a way that makes it easier for them to learn is not new: it is certainly explicit in Gray and Tall (1994).

What I want to say is based on an ancient distinction between episodic and semantic memory. This distinction, and its psychological ramifications, has been made much clearer in recent times through the work of Endel Tulving (Tulving, 1983). Tulving's early view of the distinction between episodic and semantic memory is summed up by him as follows:

"Episodic memory is concerned with unique, concrete, personal experiences dated in the rememberer's past; semantic memory refers to a person's abstract, timeless knowledge of the world that he shares with others. Distinctions of this kind had been quite familiar to philosophers interested in problems of memory, but their implications for the psychological study of memory had not been explored. ..... Episodic memory, I suggested is a system that receives and stores information about temporally dated episodes or events, and temporal-spatial relations among them. .... Semantic memory, I suggested, 'is the memory necessary for the use of language. It is a mental thesaurus, organized knowledge a person possesses about words and other verbal symbols, their meanings and referents, about relations among them, and about rules, formulas and algorithms for the manipulation of the symbols, concepts and relations'." (Tulving, 1983, pp.v, 21)

On the face of it, semantic memory sounds just like the sort of memory that is applicable to learning mathematics. Indeed, without such a memory system that was highly developed it is difficult to imagine how we could ever learn mathematics. Epsiodic memory, on the other hand, is what gives our individual lives continuity, connectedness and a sense of reality. Does this mean that mathematics is forever assigned to a part of our memories that does not deal with such matters, and must always therefore be seen as a "timeless other", existing somewhere out there as a system of codified rules and procedures? The answer is of course "no". One only has to listen to a creative mathematician talk about their work to realise that this is not how they see mathematics. Yet for most people this *is* their view of mathematics: a rule-driven, bloodless, passionless, activity, situated nowhere in time or space.

How can it be that mathematicians have such a different view of their activities from the general populace's view? One answer, I believe, is that people who learn mathematics relatively easily, who enjoy it with a passion, are in fact re-membering it through episodic memory as well as through their semantic memory system. For such people mathematics does not have to made meaningful or "real-world": it is already alive because it is situated in time and space in their lived experience in the world.

Hiebert and Lefevre (1986) use Tulving's episodic/semantic distinction in the context of conceptual and procedural knowledge in mathematics. They say, rightly in my opinion, that "The distinction between conceptual and procedural knowledge that we elaborate ... is not synonymous with any of these distinctions, but it draws upon all of them." (p. 1). The episodic/semantic distinction is therefore not to be equated simply with a conceptual/procedural distinction, although there are parallels. It is, for example, entirely possible to recall semantically whilst thinking conceptually. Indeed Tulvings explanation of semantic memory certainly involves relations between symbols and their referents.

Tulving (1983) states quite clearly that: "the absence of phenomenal experience, or concepts corresponding to it, characterizes just about every conceptual account of memory that we have had in experimental psychology, although paradoxically it is particularly true of cognitive theories of memory." (p. 125)

In the psychological study of the learning of mathematics this is a factor that seems to me to play a pivotal role: the phenomenal experience of the student engaging in mathematical activity. Let me give an example that I hope will elucidate my main point.

I asked several mathematics staff and graduate students from La Trobe University what comes to mind when they think of eigenvalues. The responses I got were typically as follows:

* Contraction maps or expanding maps.

* Straight lines being stretched; the Greek letter λ; Jordan decomposition; some sort of operation on some sort of space.

* the German language; $(n\pi)^2$ (the eigenvalues of the differential operator $Ly = y''$); the symbols λ, μ, Λ

*I imagine some sense of the "size" of a matrix related to the determinant.

* Why are we doing this? (*memory of lectures as an undergraduate*). Then reading Halmos' book and finding he called them something else.

* Characteristic numbers, Chern-Weil characteristic classes, cobordism, Smale and Milnor; Jordan canonical form, change of field of scalars, matrices with entries in non-commutative rings: it is ad-bc, not da-cb; Anosov, exp,

hyperbolicity, chaos.

* I think of numbers representing the simple "rescaling" of an object after some operation has been performed on that object.

Contrast these responses with those of a number of third year mathematics students in the same university. These students were listening to a fellow student explain his reasoning about symmetries of a cube. At one point he introduced eigenvalues and several students stopped him, apparently puzzled, asking what eigenvalues had to do with the problem. After questioning by me the entire class admitted that, for them, eigenvalues weren't *about* anything - one simply calculated them. Their memory of eigenvalues - what they re-call - is indeed semantic in the narrowest sense. There was nothing, apparently, in their memory that situated eigenvalues in time and place with a sense of meaning. They may well have been able to remember that it was a hot spring morning in a particular lecture room when they first encountered the notion explicitly, but that is a memory of themselves, of their situatedness, not episodic memory of eigenvalues in the sense that the La Trobe mathematicians exhibited it.

How is it then that some people have apparent episodic memory for what appear to be semantic events? How is it that the highly algorithmic, abstract, timeless world of mathematics is for them alive, highly connected, with a sense of continuity and situatedness in both time and space?

At first glance it seems that any one person would have to use episodic memory in much the same way as anyone else: episodic memory relates to where and when, whereas semantic memory relates to how, why, and in relation to what. However, there is another factor that is important in the distinction betwen episodic and semantic memeory, and it is the notion of context. Episodic memory is highly context dependent. In some sense this is to what the "when" and "where" of episodic memeory refers. To the contrary, semantic memory is highly context independent. It is "knowledge about" something. That the earth attracts massive objects to it, is (to the best of our understanding) not dependent upon when and where those objects are. What I suspect is that to a person who learns mathematics easily and becomes good at it, the mathematical "knowledge" so obtained retains within it elements of context. The re-call, for example, of a linear map, is for such people, not one of a definition such as $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$ (although the ability to call to mind such semantic knowledge is essential), rather it is something like a mental picture of a reflection in the plane or in space, or a rotation about a fixed axis in space, or something of that nature. These re-membered mathematical objects appear contextually, as if they were genuine objects: in that sense their re-call has an aspect of episodic memory to it. This is in contrast to what Furlong (1951) says about the difference between remembering seeing someone post a letter and remembering the square root of a number:

> "In the former case the mind looks back to a past event: we recollect, reminisce, retrospect; there is imagery. In the latter case this looking back is absent, and there is little or no imagery. We have retained a piece of information; that is all. There is retentiveness but not retrospection." (p. 6)

This characterizes for me the essential difference between a mathematical and non-mathematical bent of mind. A mathematical mind has a rich, and continually increasing, set of images associated with, for instance, the square root of a number. These images might include a right triangle, a ruler and compass construction, a rapidly converging sequence of rational numbers obtained from Newton's algorithm, A Dedekind section of the rational numbers, a construction of an algebraic number field from an irreducible polynomial, a continued fraction, a Conway game, and more. It is an imageless, decontextualized notion of something as basic as a square root that prevents many people from learning mathematics. It is as if they have decided to cut their own lively, creative mathematical actions out of their memory. But this is a mistaken notion because a person who re-members a square root in such an imageless way surely never had a lively creative action that related to square roots! Such students cannot remember relatively episodically because the only thing for them to re-member - to build again in their minds - is the dis-embodied, decontextualized written sign for a square root.

Once again I asked the La Trobe University staff and graduate students to tell me what comes to mind when they think of the square root of 2. A selection of their answers below shows the rich diverstiy of contextualized memory, including significant examples of imagery:

* $\sqrt{2}$; 1.414; Pythagoras; Samos in Greece.

* 1.414; right triangle of short side lengths 1.

* I imagine a square whose area is 2 such that each edge has length $\sqrt{2}$.

* The proof that it's irrational: fractions in lowest forms; polynomials; geometry.

* I think of a number between 1 and 2 with an infinite non-repeating decimal representation. The latter I tend to think of in terms of rational approximants to the number.

* I think of $\sqrt{2}$ as the number x such that $x^2 = 2$ and also that 1.4142 is a rational approximation. The right-angled triangle 1,1,$\sqrt{2}$ also comes to mind.

* Right triangles, scalar product, Riemannian metrics, curved spaces; $Q(\sqrt{2})$ is finite-dimensional: life is easy here.

The episodic/semantic distinction in memeory orients us to an individual's actions and their recollection by that individual. Indeed Tulving (1989) quotes Clarapede (1911) as saying that there are two sorts of memory: that relating to representations, and that relating to representations and the self. This is a central issue for mathematics learning: the objects and procedures of mathematics are not remembered in a purely semantic way by good mathematicians - they are remembered in relation to the self. And this memory in relation to the self is a memory of the self's *actions*. So, it would seem to be easy to allow anyone to be good at mathematics: promote the constructions of individuals and the re-collection of those constructions, as people do in constructivist-oriented classrooms (Martin, Pateman and Higa, 1993; Martin and Pateman, 1993; Pateman and Higa, 1993; Steffe and D'Ambrosio, 1995).

However, most of us have been through fairly dry classes in mathematics. Why do some of us come out with a passion for the subject, and and a relative ease in learning it, whilst others do not? We may ask: why do some people actively seek to experiment in mathematics, and why are they so easily able to episodically recall those experiments? I remember once telling my colleague Andrew Waywood - then my student - that one could get a better feeling for the shape of a function, for its derivative, if one imagined placing oneself on the function and sliding along it. This gives me a heightened feeling for why exponential functions are so different to monomials, and just how peculiar is the toplogist's sine curve - the graph of $sin(\frac{1}{x})$. A former colleague of mine, Ken Miles, could solve complicated topological problems in his head: he confessed to having continually moving images, even when he was thinking of algebraic formulas which he would see weaving and dancing. A grade 3 student whom Robert Hunting and I worked with several years ago showed, after a short time, exceptional ability in rational number problems. He confessed to us that he habitually split numbers into their parts - found their divisors, in other words. This mental activity, constantly practised and recalled, allowed him to easily evaluate complicated rational number comparisons: such as whether $\frac{3}{5}$ was bigger than $\frac{5}{8}$ and by how much. So now let me ask: does the process of putting life into mathematics, of being able to re-call mathematics in a contextualized wat, with episodic features, have to do with purposeful, intentional activity in mathematical settings? The answer, I believe, will turn out to be "yes". How we might find this out is by doing PET and MRI studies of people thinking about mathematics. If intentional brain centres can be isolated by these techniques there is also a possibility that we can distinguish episodic and semantic areas, and so relate different levels of mathematical achievement to differing modes of thought.

For the reasons I have outlined, I am pessimistic about a majority of people ever being able to learn mathematics effectively. It is not that most people *could*

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not understand mathematics relatively easily. The problem is, I believe, that in order to recall mathematics so as to be able to do it easily they have to recall it episodically. But in order to do that there must be episodes and contexts to recall. These episodes and contexts require active construction by individuals. Why, however, should a young student's mind actively engage with mathematics when they don't yet know what it is about? It is not, after all, an everyday subject that most people talk about as they do the weather, their health, or the activities of their friends and neighbours. But perhaps it could be: maybe it could be as lively a reminiscence in one's mind remembering the square root of 2 as remembering someone post a letter.

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THE ILLUSION OF LINEARITY: A PERSISTENT OBSTACLE IN PUPILS' THINKING ABOUT PROBLEMS INVOLVING LENGTH AND AREA OF SIMILAR PLANE FIGURES

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Abstract. *Linear (proportional) functions are undoubtedly one of the most common models for representing and solving both pure and applied problems in elementary mathematics education. But according to several authors, different aspects of the current culture and practice of school mathematics develop in pupils a tendency to use these linear models also in situations in which they are not applicable. This paper reports an exploratory study about the occurrence of this "illusion of linearity" in 12-13-years-old pupils working on word problems involving length and area of similar plane figures of different kinds of shapes, as well as about the influence of drawings in breaking the linearity illusion for that kind of geometrical problems. The results provide a convincing demonstration of the strength of the illusion of linearity among these pupils.*

Introduction

Linear relationships are a major topic of elementary mathematics education. For a lot of problems in pure and applied mathematics, linear functions are the underlying mathematical model. Unfortunately, pupils' growing familiarity and experience with linear models may have a serious drawback: it may lead to the misbelief that these models have a "universal" applicability and to a tendency to apply them for representing and solving problems which cannot be properly modelled in terms of a linear proportional function. This phenomenon is sometimes referred to as the "illusion of linearity". Although this linearity illusion has been described and illustrated by several authors (Berté, 1993; Freudenthal, 1987; Rouche, 1989), it has elicited little or no systematic empirical research.

The examples of the illusion of linearity provided in the above-mentioned references relate to different domains of mathematics, such as algebra, geometry, probability and statistics. In the present study we focus on mathematical applications dealing with lengths and areas of two similar geometrical figures. The principle governing that kind of application problems is well-known: an enlargement or reduction by factor ρ, multiplies lengths by factor ρ, areas by factor ρ^2 and volumes by factor ρ^3. These factors depend only on the magnitudes involved (length, area, and/or volume), and not on the particularities of the figures (whether these figures are squares, circles, etc.). Freudenthal (1983) argues that there is hardly any insight that is as fundamental for the constitution of these mathematical objects and their measurement.

The acquisition of this principle is a difficult process. The literature contains several illustrations of misconceptions regarding the different behaviour of one-, two- and three-dimensional magnitudes under enlarging and reduction operations. The most famous example can be found in Plato's dialogue "Meno", in which a slave, when asked by Socrates, Plato's master, to draw a square having two times the area of a given square, firstly proposes to *double the side* of that square. In the present study, we investigate more systematically the illusion of linearity in 12-13-years-old pupils working on word problems involving length and area of similar plane figures of different kinds of shapes, as well as about the influence of drawings in breaking this illusion for that kind of geometrical problems in pupils' reasoning about length and area of similar plane figures.

Method

Hundred and twenty 12-13-year-old pupils from 6 different 7th-grade classes of a large secondary school in Flanders, recruiting its pupils from a wide diversity of elementary schools in the region, participated in the experiment. These 120 7th-graders were divided in three equal groups of 40 pupils each; each group consisted of two intact classes.

The experiment consisted of two phases. In the first phase, all pupils were administered the same paper-and-pencil test consisting of 12 experimental items and some buffer items. All 12 experimental items involved enlargements of similar plane figures, and belonged to three categories: 4 items about squares (S), 4 about circles (C), and 4 about irregular figures such as maps (I). Within each category of figures (S, C, and I), there were 2 proportional and 2 non-proportional items. Table 1 lists an example of each: a proportional (item 1) and a non-proportional (item 2), both dealing with squares.

1 Farmer August needs 4 days to dig a ditch around his square pasture with side 100 m. How many days will he need to dig a ditch around a square pasture with side 300 m? *(Answer: 12 days)*

2 Farmer Carl needs 8 hours to manure his square piece of land with side 200 m. How many hours will he need to manure a square piece of land with side 600 m? *(Answer: 72 hours)*

Table 1. Two examples of experimental items about squares

As illustrated in Table 1, the variables "length" and "area" were sometimes replaced by more concrete variables that are directly proportional to them (or that can reasonably supposed to be so), with a view to construct a set of meaningful application problems.

Two weeks later the three groups of pupils were confronted with a parallel version of Test 1. The problems in this second test were the same for all three groups, but the instructions were different. In Group I, which functioned as a control group, the testing conditions were exactly the same as during the first test, which means that these pupils received the problems without any additional hints or instructions. The pupils of Group II were explicitly instructed to make a drawing or a sketch of the problem situation before computing their answer. This instruction was given at the beginning of the test by one of the researchers, and was illustrated by means of an example item (which did, of course, not involve similar plane figures). In Group III, every problem was accompanied by a correct ready-made drawing like the one given in Figure 1 (which corresponds to item 2 in Table 1).

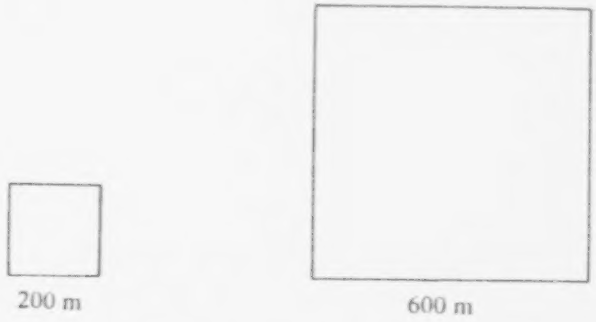

Figure 1. Ready-made drawing corresponding to item 2 in Table 1

Hypotheses

First, on the basis of the available literature (Berté, 1993; Freudenthal, 1983; Rouche, 1989), we hypothesized that the vast majority of the pupils would suffer from the illusion of linearity, and that they would therefore use a linear model to represent and solve not only the proportional items but also the non-proportional items. Consequently, we predicted that the pupils' performance on the proportional items would be very high, while their scores on the non-proportional items would be very low.

Second, we assumed that the drawings would have a beneficial effect on the pupils' performance. More specifically, it was hypothesized that asking pupils to generate a drawing before performing any computation would stimulate them to construct a proper (mental) representation of the essential elements and relations involved in the problem, especially for the non-proportional items; this representational activity would help them to detect the inappropriateness of a stereotyped solution based on linear proportional reasoning, and to determine the nature of the non-linear relationship connecting the known and the unknown elements in this problem representation. Furthermore, we hypothesized that giving pupils a *correct* drawing of the problem situation would even be more effective than instructing them to generate such a drawing, because in the latter case pupils may not succeed in making a *correct* drawing themselves. Starting from these hypotheses the following predictions were made: in Group I the results will remain the same for Test 1 and Test 2; in Group II the percentage of correct answers will increase from Test 1 to Test 2; in Group III this increase will be greater than in Group II. Moreover, we anticipated that the higher performances on Test 2 in Group II and III would be essentially due to a decrease of inappropriate solutions based on proportional reasoning on the non-proportional items during Test 2.

Third, we predicted that the pupils' performances would be different for the distinct types of plane figures involved in the study. More specifically, the items about squares (S-items) were supposed to be the easiest and those about the irregular figures (I-items) the most difficult. We also expected that the size of the anticipated effect of the availability of drawings (see hypothesis 2) would be affected by the type of figure, in the sense that this drawing effect would be the greatest for the S-items and the lowest for the I-items. The rationale behind these predictions is exemplarily worked out for the non-proportional item in Table 1. To find the answer to this item, the problem solver can choose among three appropriate solution strategies: (1) paving the big square with 9 little squares, (2) calculating the area by the formula "area = side × side", and (3) applying the general principle ("if length × ρ, then area × ρ^2 "). For the variant of that item about circles (the C-variant), the first solution strategy becomes impossible and the second strategy becomes more error-prone (because of the greater complexity and unfamiliarity of the formula for calculating the area), and the variant dealing with irregular figures (the I-variant) can only be solved by means of the general principle.

Results with respect to the hypotheses

Table 2 gives an overview of the percentage of correct answers of the three groups of pupils (I, II, and III) for the proportional and the non-proportional problems about squares (S), circles (C), and irregular figures (I) during Test 1 and Test 2.

	Test 1						Test 2					
	proportional items			non-proportional items			proportional items			non-proportional items		
	S	C	I	S	C	I	S	C	I	S	C	I
I	96	98	89	5	0	1	99	96	85	3	0	0
II	93	95	89	6	1	0	93	95	95	4	2	0
III	91	91	87	4	3	0	93	89	89	8	5	1

Table 2. Overview of the results

The results provide a very strong confirmation for the first hypothesis. Indeed, an analysis of variance revealed an extremely strong main effect of the task variable "(non-)proportionality" ($p < .01$): while the proportional items elicited a very high overall percentage of correct responses, this percentage was extremely low for the non-proportional items: for the three groups and the two tests together, the percentages of correct responses for all proportional and for all non-proportional items were 92 % and 2 %, respectively.

Second, the results did not support the second hypothesis concerning the effect of the drawings on the pupils' performance. The analysis of variance revealed no significant "Group × Test" interaction effect: for none of the three groups there was a significant increase in the pupils' scores from Test 1 to Test 2 in general, and in their performance on the non-proportional problems in particular. For Group I the percentage of correct responses on the non-proportional items decreased slightly from 2 % to 1 % between Test 1 and Test 2; for Group II the percentage remained the same at 2 % during both tests. And for Group III, the percentage of correct answers on the non-proportional items increased slightly between Test 1 to Test 2, but remained still extremely low during the latter test: from 2 % to 5 %. In sum, while there was a trend in the expected direction, the obtained differences were too small to be significant. Apparently, the anticipated positive impact of the instruction to make drawings (in Group II) and of the provision of these drawings (in Group III) was too weak to break the overpowering linearity illusion among these 12-13-years-olds.

In line with the third hypothesis, the type of figure had a significant effect. The analysis of variance revealed a significant main effect for this task variable ($p < .01$). The scores for the S-, the C-, and the I-items were in the expected direction - the overall percentages of correct answers for these three kinds of problems were 49 %, 48 %, and 45 %, respectively -, but additional Tukey tests revealed that only the difference between the I-items and the S-items and between the I-items and the C-items were significant (both at the 1 % level). Moreover, no interaction effect was found between the task variables "(Non)proportionality" and "Figure", which indicates that the observed main effect of the type of figure was found in the proportional items as well as in the non-proportional items.

Additional findings

After presenting the quantitative results with respect to the research hypotheses, we also briefly discuss some qualitative findings based on a systematic and fine-grained analysis of the notes written down on the pupils' response sheets, which may help to explain these quantitative results.

First, the qualitative inspection of the notes on the response sheets of Test 1 (and also of the notes of Group I during Test 2) revealed that pupils very rarely spontaneously constructed a drawing or a sketch of the experimental problems in general and of the non-proportional problems in particular. It remains unclear whether the remarkably small number of spontaneous drawings on the non-proportional problems was due to the fact that the pupils were not capable of applying this valuable heuristic after they had detected the problematic nature of these non-proportional problems, or to the fact that they did not even think of applying it because they never remarked the unfamiliar and problematic nature of these items.

Second, the inspection of the notes on Test 2 in Group II revealed that - in spite of the explicit instruction to do so - the vast majority of the pupils did not produce drawings. This finding may explain why there was no increase from Test 1 to Test 2 in Group II. Again, it is unclear whether these pupils did not follow the instruction to make a drawing because of their incapability to apply this valuable heuristic in a genuine problem situation, or simply because they saw no reason to make a drawing for problems which they considered as easy and familiar. Although the notes of the pupils from Group III on Test 2 provide no *direct* information about the extent to which the ready-made drawings were effectively used by these pupils, the virgin figures on their notes suggest that most of them paid little or no attention to them, probably because they approached the tasks with the expectation that all items could be solved in a routine-based way by means of linear proportional reasoning.

Finally, the above-mentioned qualitative findings do not imply that the pupils' notes consisted merely of one or two multiplicative number sentences. On the contrary, the majority of the pupils made more or less systematically use of a so-called proportionality table, involving an abstract representation of the (assumed) linear proportional relationships between the known and the unknown elements by means of arrow diagrams. In Flanders, pupils of the upper years of the elementary school are intensively trained to apply such arrow diagrams for representing and solving word problems about ratio and proportion. However, the results of the present study suggest that for these pupils the schematisations have completely lost their original function as a valuable modelling and problem-solving tool. Rather than stimulating and helping pupils to analyze complex problem situations for which they have no ready-made solution, they have become a part of a mindless, ritual activity in which the pupil immediately sets up and works out the proportionality schema that "fits" the problem, without any critical consideration of the appropriateness of the mathematical model underlying their stereotyped solution.

Conclusion

The present study convincingly demonstrates the strength of the illusion of linearity with respect to problems involving lengths and area of similar plane figures in 12-13-years-old pupils. However, further research is required to provide a more complete and systematic picture of the occurrence of this illusion, both in terms of problem types and of educational levels. Moreover, there is a need of a better insight into the conceptions and strategies underlying both the correct and incorrect solutions of linear and non-linear problem situations by means of more process-oriented research. Finally, future research should aim at unravelling the instructional factors which foster the occurrence of the illusion of linearity as well as at developing and testing new instructional materials and techniques for preventing and/or remedying it. In this respect, the results of the present study suggest that too much attention is paid at training pupils in the computational aspects of ratio and proportion and in the use of stereotyped representations of linear proportional relationships, while the appropriateness of a linear proportional model in a given situation is seldom discussed.

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The development of whole-part relationship in situations of combine structure for a student with learning disabilities

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Abstract

The purpose of this research was to expand the knowledge with regards to the constitution of an understanding of the part-whole relation for a child with learning disabilities. The analysis of initial mental representations, procedures and reflections that arise allow us to appreciate the evolution of the child's thinking process. Hence, for this child, when situations are conducted to illustrate the quantities in question, the counting process involved suscitate a reflection over relations vis-à-vis quantities which further elaborate the part-whole relation. However, this coordination is not sufficient. It is essential to recognize simultaneously the distinct and uniting properties of the sub-subsets.

Le but de cette étude vise l'amélioration des connaissances eu égard à la construction d'une compréhension de la relation parties-tout chez une enfant en difficulté d'apprentissage. L'étude de ses représentations initiales, de ses procédures et des réflexions qui émergent nous permettent d'en apprécier l'évolution. Ainsi, pour l'enfant rencontré, lorsque la situation proposée l'amène à illustrer les quantités en jeu, les procédures de comptage qu'elle y coordonne suscitent une réflexion sur les relations entre les quantités, ce qui facilite l'émergence de la relation d'inclusion. Toutefois, cette coordination n'est pas suffisante. Il est essentiel qu'elle reconnaisse les propriétés, à la fois, distinctes et unissantes des sous-ensembles.

Introduction

This exploratory study concerning word problems involving addition and subtraction had a combined structure. In fact, while conducting a research on our system of symbolization of numbers (DeBlois, à venir), we realized the impact of this type of relation with regards to the development of the understanding of this concept. We investigated further, our investigation on the formation of this logical relation, considered fundamental (Inhelder and Piaget, 1967). We attempt to determine how day to day experiences of combined structures would influence the constitution of an understanding of this type of relation. We could then recognizing the contribution of these experiences in the understanding of the child.

1.0 Frame theory

The intention is to approach this question, taking into account the child first, which places the child in a primary position: its way of learning to evoke its knowledge and to select reasonning procedures leading to reflections which illustrates flexibility of the mind (Gray and Tall, 1994; Minskaya, 1975).

1.1 Reflective abstraction

Piaget's reflective abstraction model explains the construction of an understanding, called "reflection". This reflection permits us to restructure the equilibration process. Starting by assimilating the information, which could be disturbing, and the accommodation of knowledge to a proposed situation, new coordinations, between representations and regulations of the child, suscitate a reflection. Thus, we tried to extract the initial representations evoked by the child with learning disabilities, the resolving procedures (used as regulations) and the coordinations realised between us. In fact, the initial mental representations and the procedures are the baseline to build reflection.

1.2 Initial mental representations , procedures and reflections

Initial representations of the child are influenced by the environment, their experiences and their abilities to learn. Their initial mental representations are the "refléchissement" of actions on the mind (Piaget, 1977). They permit to expand to new constructions. Initial mental representations seem to be one part of the learning process which allow us to understand the construction realised, by the child. While being an important aspect, this is not the only element to consider when studying the learning process of the child who has learning disabilities.

Many researchers observed the procedures used by the child to resolve additions and subtraction. The study of Carpenter, Hierbert and Moser (1979), Carpenter and Moser (1982), Case (1982) each described many different counting procedures that were not taught to students. Gray and Tall (1994), in their research on the duality of processes and the concept represented by the same symbols, found that the utilization of procedures where numbers are, simultaneoustly the object and the process would be the principal success factor. Cases where this appeared as a fixed counting procedure would be one of the principal failing factors. A flexible approach of the mind would be the key linking procedures and concept.

Children build up different reflections like commutativity or associativity. We will concentrate specifically on the inclusion relation, also referred to as the part-whole relationship. Our research question will therefore be: *What are the initial mental representations and the procedures coordinated by a child with learning disabilities regarding the part-whole relation?*

2.0 Method

During the school year, as part of the standard practice, teachers of regular classrooms will be requesting the assistance of specialized teachers called "orthopedagogue". This type of assistance is required when a child shows persistent learning disabilities. This will be the starting point of our research.

Inspired by clinical interviews of Piaget, we tried to determine the counting abilities of children, their understanding of symbolization numbers, addition and subtraction operations using specific evaluation material developped for orthopedagogues. (Jolin, DeBlois and Roy, 1993).

To prepare intervention, we are inspired by teaching experiment (Steffe, 1983). We élaborate our teaching experiment using as a baseline the model of the development of understanding developped by Bergeron and Herscovics (1989a). In this model, the authors attempt to describe the components involved in the elaboration of an understanding of a concept. They insisted on the presence of comprehension on the part of the child well before he or she learns to use mathematical symbols specific to a particular concept, such as a number or an addition. This is what lead to the appearance of the two-tiered model. The tier of preliminary physical concepts describes an intuitive procedural and abstraction understanding. The level of emerging concept illustrates the interactions which could happen between the other components: procedural, abstraction and formalization. They already proposed a conceptual analysis of addition. To do so, we developped criteria allowing to realize an analysis of situations involving subtractions. Then, we also explore four situations to the maximum.

Many researchers demonstrated the impact of a word problems formulation versus the success of children (Rosenthal and Resnick, 1974; Fayol and Abdi, 1986; Fayol, Abdi and Gombert, 1987). We already know that when the representation of a situation described appears to be an easy simulated action (Fayol, 1990), young children obtain success. Nesher (1982) said that many structural variables such as logic and linguistic, syntax and semantic components tend to produce an effect where problems will appear more, or sometime less comprehensible. The proposed situations in this study permitted us to observe how a child, who has learning disabilities, can resolve word problems on discrete quantities. The content of each situations will refer to objets that are, or have been, manipulated in the course of a regular child's activities. It will introduce natural numbers that can be counted by the child. We proposed the following situations during our teaching experiment.

1. This autumn, you went to pick apples in an orchard. You picked 420 apples. Simon picked 260 apples. Carmen picked 315 apples. All apples were placed in a bag. How many apples are there in the bag?
2. There are 18 fruits in a basket. You see 6 apples. The other items are kiwis. How many kiwis are there?
3. You want to have red and green pens. You will have a total of 38 pens. You will buy 12 red pens. How many green pens will you buy?
4. You have organized an afternoon of games for the birthday party of your best friend. You phone his cousins. You also phone 6 of his classmates. In total, you called 13 children. How many cousins did you call?

Before these experimentations, we met the orthopedagogue who realised these experiments. She had to be familiarized with our frame theory. The researcher continously kept in touch with the orthopedagogue to support any interventions and answer any questions. Interviews were taped and transcribe before analysis.

3.0 Preliminary results

3.1. Initial evaluation of Karoline

Karoline is a 9 years old girl. Our evaluation interview allowed us to observe her abilities of counting, the understanding of symbolization numbers and finally of the addition and subtraction operations. In the course of the interview, Karoline seemed to use the numbers without being assigned an outside quantity idea.

Thus, her counting abilities were rudimentary. For example, she had difficulty to keep regularity when countig by 2, by 5 and by 10 and backwards. The transition to another hundred increased the difficulty level. She accurately wrote the numbers asked for, found the numbers directly placed before or after and decomposed easily, the number 357. However, when questionned regarding the meaning of the symbols used, difficulty became evident. She counted 14 set of ten tokens and 2 single tokens: "10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, 400, 142". On the other hand, when the organization of the tokens were modified, she did not recognize the invariability of the quantity. Finally, she believes that the horizontal addition of 156+78 gave a different sum from that when the said numbers are placed verticaly. Also, to verify the results of the operation 89 - 42 she believed ones could do the opposite operation which would be 42 - 89. The same type of pattern was presented with the following situation 9-7=2, without better results. She then added 7+9=16 and found it did not work. A mystic wall still remained between the formalization and the process subject of this experiment. These difficulties could contribute to deepen the gap between procedures and formalization.

Finally, in situations where addition and subtraction operations are called for, she demonstrated a predominance of the mathematical symbols used whitout the quantity idea. Rather than selecting the two quantities asked for, she gathered all the numbers to add them up without considering relations that can intervene in those situations. However, when placed in a word-problem were the numbers are smaller than 10 (10 and 3), she recognized the relations between quantities.

3. 2 Teaching Experiments

The first situation allow to identify a representation based on quantity of numbers. Hence, Karoline believes at first glance that her bag held more apples than the bigger bag. Then she realized that there are more apples in the bigger bag because there are more numbers. She is then asked the quantity of apples she will have in her big bag if she was to remove Simon's 265 apples and Carmen's 315 apples. Karoline tried to perform a subtraction. This was for her a difficult task. After having removed Simon's 265 apples, she first tried to remove 315 from the original quantity (1000). When she got to the remaining 420, she does not recognize that it was in fact **her** quantity of apples.

L'orthopédagogue proposed a similar problem with the numbers 10, 8 and 12. She found the total quantity and could remove the families of apples one after another to find the delta every time. She discovered that by considering first the quantities of apples of each individuals, she didn't have to calculate. She then drew a set representing her 420 apples, Simon's 265 and Carmen's 315. She easily found the remaining apples when first removing the apples of the others respectively.

Situations 2 and 3, where the set and one of its subset is known, lead to discover other initial representations and also other procedures. Karoline showed the total quantity with 18 small cubes. When asked to retrace the 6 apples among the 18 fruits, she hesitated and reviewed the situation. Then she counted 6 apples among the 18 fruits and then counted by one the other portion to obtain the 12 kiwis. When asked if there were more apples or more kiwis, she believed that the quantities were equal. Then she explained: "There are more apples because there was 18 initially". She was then requested to show the apples and the kiwis and the fruits. She showed the apples and the kiwis, and then explained that the fruits: "Are not there. There gone". The orthopedagogue asked the child to show the apples and the kiwis. She recognized subsequently the 18 fruits. She established that there are more kiwis than apples and that there are more fruits than kiwis. However, she mentionned: "The apples, there are only 6 and the fruits, there are 12". After being told that apples and kiwis are fruits, she understood that there are more than apples: "Because the apples are also fruits". She explained subsequently that to discover the quantity of kiwis, one must remove the quantity of apples.

The situation is subsequently inverted. You have 18 fruits in a basket. There are apples and kiwis. You count 12 kiwis. How many apples will you have? She repeted: "Apples? You have 18 fruits in a basket. It contains apples. It contains apples and kiwis. You count 12 kiwis. How many apples? I'll remove some". She took back the 18 cubes, took out 12 kiwis, found 6 apples then established that she did not have to count. Karoline observed subsequently that for both situations: "This was not equal but it was the same drawing. ... They changed the numbers".

When we present the third situation, Karoline takes 38 sticks to represent 38 pencils. However, subsequently she tells the story many times and hesitates. We then wrote on a sheet of paper 12 red pencils and 38 coloured pencils. She stops for a while then explains some are missing. She takes the sheet and writes numbers from 12 to 38, counted those written numbers and found 26. She completed the equation 12+26=38. She now knows that there are more green pencils. There are only 12 red pencils and we must add up to 38. We asked her if she could have proceeded differently. Counting with my fingers would have been difficult, she said. She explained also that she could have calculated 38-12=26 "since there is only one number".

The forth situation that initially showed only on part then the hole picture induces a difficulty to determine what she was looking for. She wrote 6 on her paper then 13-6=7. She explained that we don't know how many cousins, therefore a minus must be used. She found the 7 friends that were called. We asked her to review the question. She subsequently understood that the number 7 corresponded to cousins. She mentionned that as a whole, she had 13 friends, a total of 13 children. She invited 6 friends in her classroom and the rest we don't know therefore it gives 7. She recognized the inclusive relations by saying that she phoned more cousins than friends and then more children than cousins.

4. Discussion

The initial evaluation allowed us to know that numbers did not represent a mathematical objet and a procedure. We determined, at this time, there was a gap between procedure and formalization. These situations allow her to reflect about quantity, reflection, issued by own actions. These reflections contribute to the development of a hierarchial structure about set and subset. Nevertheless, when she used numbers like physical objets, this structure did not appear.

During the teaching experiments, these initial representations allow us to observe that numbers are seen either as physical objects or as representatives of quantities. These two types of representations suscitate the insertion of various procedures: adding operations, numbering of objects and of numbers. Her understanding of inclusive relation seems to increase specifically when she considers the numbers as representatives of quantities. In fact, at that particular time, she coordinated counting procedures and could consider the relations between the quantities. The situations of complement were interesting for this. Karoline was obliged to compare between subset, and after a subset, with the total set. Subsequently, she can talk about whole-part relationship.

In conclusion, the large numbers but also the difficulty to recognize simultaneously the distinct and uniting properties of the set and-subsets prevented the utilisation of the part-whole relationship in day to day situations. However, to help Karoline to introduce this type of relation in her solution, it were important to illustrate numbers, to find the total set and to use known facts.

5. Implications for research

In this case study, Karoline saw numbers like physical objects, then physical entities, irresolvable. These initial mental representation of numbers did not incite to flexible thinking. How children adapt this part-whole relationship for different concept like symbolization numbers or combine structure of word problems? Is it possible to present situations that could facilitate the passage between different concepts?

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STUDY ON THE EVOLUTIÓN OF GRADUATE STUDENTS' CONCEPT IMAGES WHILE LEARNING THE NOTIONS OF LIMIT AND CONTINUITY

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Abstract

This is a report about a case dealing with the evolution of students' concept images as related to the definitions of Limit and Continuity. The theoretical ontogenetic-socio-cultural and pedagogical framework defines analytical units in order to scrutinize the student's development and concept image as related to the initial and potential cognitive stage. Teaching situations are used to originate cognitive conflicts, and the evolution of concept schemes during the process of teaching-learning is analysed. Conclusion is reached which tell us of the feasibility of setting out pedagogical situations which allow for controlling the conceptual change and for an efficient relationship between learning to cognitive development.

1. Introduction

Research hereby presented is linked to the pursuit started in 1986, at the Universidad del Valle (Cali, Colombia). A teaching-learning strategy was devised and applied for Calculus I course, aiming at achieving a comprehensive learning of concepts. The aim was to have the target subject establish relationships between the new knowledge object and elements already established inside his cognitive structure, all done in an essential and coherent manner. This pursuit led to Delgado's (1995) research, subject-matter of this report. The *intended problem to be investigated* is the following:

- In a learning situation, is it possible to control the cognitive structure evolution of the student, as related to conceptual difficulties and to cognitive hindrances arising in connection with the notions of Limit and Continuity?

- Is it possible for a pedagogical sequence where conceptual difficulties and cognitive hindrances are taken into account, to be efficient enough as to boost the evolution of cognitive structure related to those notions?

We cannot find any study facing this problem within the available literature.

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Research is focused on the study of written outcomes from a student considered as a member of a group. These outcomes are structured according to a guide acting as a booster for situations capable to provoke cognitive conflicts. This conflicts are generated, regulated and strengthened by means of the *intercommunicating action* of debate in the classroom (interpersonal processes) and are internalized by means of the individual *cognitive processes*: assimilation, accommodation and balancing (inner-personal processes). Activities, as much as objects and methods, are common for the group, and the group states its requirements and *regulates actions*, taking into account theoretically established *basic principles*.

As a consequences of consecutive coordinations between the different concept images, the student is expected to manage encapsulating the definition of continuity inside the following terminology:

$$(\forall\varepsilon>0)(\exists\delta>0\ (|x-p|<\delta \Rightarrow |f(x)-f(p)|<\varepsilon\ )).$$

2. Theoretical Framework

Works by Piaget (1981) provide with elements of analysis for the study of conceptual change: structure, outline and its relationship with assimilation processes, accommodation and balancing. This last process gives reasonable explanations about how and why the subject's cognitive enhancement is produced. Balancing is the mechanism which maintains cognitive organization and it appears because of disturbances caused by the subject's activity, and because of compensations the organism opposes to disturbances –achievement of various new *possible conceptions* and coordination among what is *possible*, *necessary* and *real*–. As a *process*, balancing is the factor unifying conceptual change and development –adaptation– . As a *state* (in equilibrium) it is the continuously changing balance of active compensations –selection–. This theory eases the analysis of the subject's outcomes from the stand point of inner processes (inner-personal), but ignores the external processes (inter-personal) affecting the cognitive development.

We complement the ontogenetic approach by adding Vygotsky's (1962) socio-cultural approach, while always keeping in mind Habermas' (1984) and Wertsch's (1985) updates, paying special attention to concepts like mediated action, measure instruments and close development zone: "Vygotsky argued that, starting at a specific moment of development, biological forces cannot be seen as the only, or even the main, changing force; a fundamental arrangement in development forces is brought about as well as a need for its corresponding rearrangement of principles explaining the system [...]. The center of the explanation switches from biological factors to social factors." (Wertsch 1985).

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These theories provide with analytical units in order to study conceptual change from an ontogenetic and socio-cultural standpoint. We will use the definition of "*concept image*" (C.I.) (Tall and Vinner, 1981; Azcárate, 1995) when talking about those elements of the cognitive structure put into action in connection to mathematical notions. We define *conceptual change* as the consequence of solving a conceptual problem derived from a mistaken or insufficient C.I. in front of the "ideal explanations" required to face the situation. Conceptual change comes from awareness of conflict in recognizing C.I.'s limitations to confront the need to find the "rational explanations" required by the group taking part in the teaching pursuit. This definition takes into account Toulmin's (1977) views and links socio-cultural and ontogenetic domains.

We use the above definitions, units and theories in order to study the following variables: conceptual change (selection) and balancing (selection and adaptation). As far as epistemological hindrances linked to the notions of limit and continuity are concerned, we consider Brousseau's (1983), Sierpinska's (1985) and Cornu's (1986) researches. As for the development of advanced mathematical thought, we contrast thesis by Sfard (1991) against Tall's thesis (1994, 1995), on the role of algorithms and concept formation in mathematical thought, adapting Tall's notion of "*procept*" in order to explain the strategy in constructing the (ε-δ) definition of Limit.

3. Research Method

This research consists of a case study based on a student's written outcome in a learning situation, and corresponds to the introduction of the notions of continuity and limit.

Tools used in the classroom are: Text (T. Apostol, Calculus) and a guide.
Data have been analyzed from:

- Systemic Networks (figure 1 presents whatever concerns continuity)
- Protocols: for the guide and for student's answers
- A chart of codes, where the student's reactions are recorded and coded in terms of: C.I., applications of C.I., conflicts, difficulties, epistemological hindrances and interpretations of instructions.
- A diagram of C.I. evolution, developed from the chart of codes.

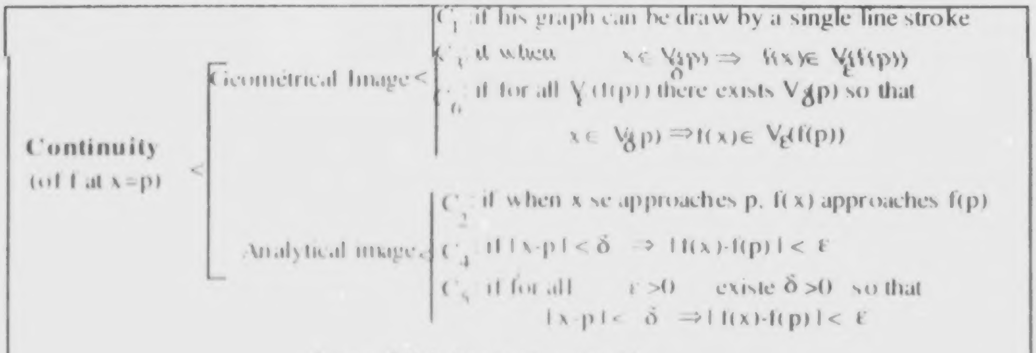
Continuity
(of f at x=p)
Geometrical Image
C1 if his graph can be draw by a single line stroke
C3 if when x ∈ Vδ(p) ⇒ f(x) ∈ Vε(f(p))
C6 if for all Vε(f(p)) there exists Vδ(p) so that
x ∈ Vδ(p) ⇒ f(x) ∈ Vε(f(p))
Analytical image
C2 if when x se approaches p, f(x) approaches f(p)
C4 if |x-p| < δ ⇒ |f(x)-f(p)| < ε
C5 if for all ε >0 existe δ >0 so that
|x-p| < δ ⇒ |f(x)-f(p)| < ε

Figure N° 1 Systemic network

The guide includes 25 situations, through which questions are submitted, and comments, hints and instructions are offered. Eg. G_1[1] shows a drawing of a ball rolling down over an inclined plane, then going on over a horizontal plane and lastly climbing up a final plane; then it asks to draw the corresponding graphs for the speed and acceleration of the mobile object and to characterize those graphs at the precise points when movement changes. Attention is drawn towards the aspect we are interested the student to define.

G1 (d) What are the speed v(t) and the acceleration a(t) graphs like at the instant t_1 when the ball crosses B? At the instant t_2 when the ball crosses C?

A mathematician would say:

- ***"Speed function is continuous at all points, while the acceleration function is not continuous at t_1 y t_2."***

G3. Write down your idea for the definition of a function's continuity at point x=p.

G5. consider the following functions: $f(x)=x$; $g(x)=\begin{cases} x & x \in Q \\ 2 & x \in I \end{cases}$ Using your own definition, determine whther: a) f is continuous at x=1 b) g is continuous at x=2
Hint: consider two situations. One where f will be continuous at x=p and other where f will not be continuous. Note what value f(x) tends to when x approaches p

Situations such as those stimulate the appearances within the group of distinct Cis (Fig.1). Debate in the classroom calls for the subject's definition to account "rationally" for every situation. Thus, in case C_1 (Fig 1) is stated in response to G_3, its transposition to situation G_5 will not withstand debate inside the group. C_1 is expected to induce a conflict in considering that function g graph cannot be drawn "in a single line stroke" even though it is continuous at x=2. Conflict

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[1] Guide's situation number i (i=1,2,3,...25) will be noted as Gi.

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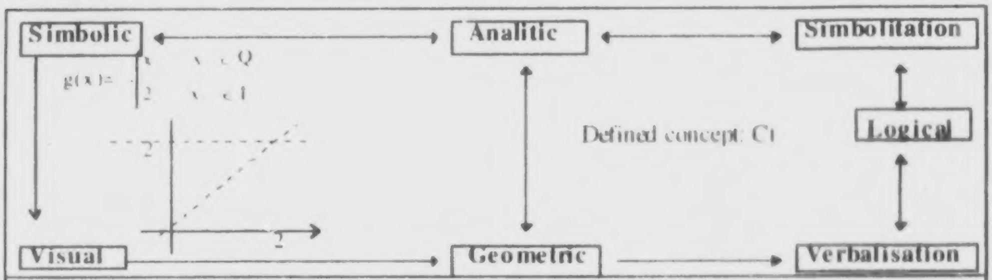
comes as a result of cognitive processes originating awareness of C_1 *global geometric* C.I. limitation to account for the situation. This should lead to construct C_2, local analytical, concerning "nearby" numerical values, only suggested in G_5 hint. We illustrate (fig. 2) the constructive process, following Tall-Piaget's theory.



Cognitive processes take part in the generation of conflict.

Assimilation: *in drawing the graph, pencil is uplifted.*

↓ ↑ ←**Conflict**→ **Reality:** *f is continuous at x=2.*

Accommodation: *if pencil is uplifted, it is not continuous.*

Conflict amplifies the world for possible:
-Response is erroneus.
-It is broken but not briskly.
-It is broken but f(x) values should be close to f(p)

Previous facts lead to construct what is need:
-Graph should not brake.
-Braking should not
-f(x) should take values close to f(p) wen x takes values close to p.

Possible, necessary and real are coordinated to counterbalance disturbance and to install equilibrium:

- Alfa equilibrium.
-If it brakes is not continuous.
-Response is closed.

- Beta equilibrium:
-when braking is not brisk, f is continuous

- Beta equilibrium:
f is continuous at x=p if, when x takes values close to p, then f(x) takes values close to f(p).

A gama equilibrium corresponding to C_5 or to C_6 is not to be expeted

figure 2

Each equilibrium state is controlled by the "rational" requirements of the group at each one of the 25 situations, thus generating a dynamics for conceptual change which can be perceived through the student's written outcomes.

4. Instance from the analysis

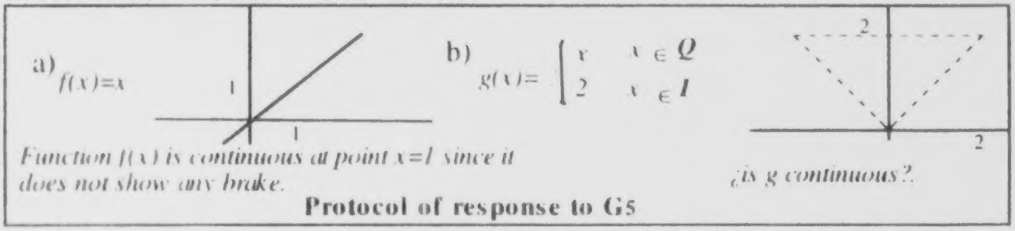
As a result of interaction with G_1 and G_2, the student writes about a G_3 the following statement

> *"F is continuous when we do not observe brisk breaks in its values. Its graph can be drawn without uplifting the pencil*

Protocol of response to G_3

This definition corresponds to C_1 (geometrical, global); C.I. is activated by the graphical context of the situation. In G_4, question is about continuity of the function "minor integer" at x=1/2 and x=1; the student applies properly his own definition C_1 for the affirmative and negative cases. This is a control question.

In relation to G_5, the student wrote:



a) $f(x)=x$
1
1
b) $g(x)=\begin{cases} x & x \in Q \\ 2 & x \in I \end{cases}$
2
2

Function f(x) is continuous at point x=1 since it does not show any brake.

¿is g continuous?.

Protocol of response to G5

(a) Draw the graph and apply C_1 properly.
(b) He does not respond. He only asks himself: is g continuous?

- Application of negative C1 is empty.
- His graph does not correspond to the one defined by the algebraic expression of g.
- The required conflict (P1) between "continuity without brake" and "continuity with brake" to evolve from C_1 to C_2 does not emerge.
- C_2 definition has not emerged.
- He has not followed No. 5 instructions.

5. Results

From analysis of the first five situations we gather:

1. The student has understood and assumed the task of constructing the definition
2. Initial C.I. corresponds to C_1 and the student does hold another C.I.
3. There exists a conceptual difficulty concerning the interpretation of the algebraic expression of a function defined by "portions"
4. The student became aware of C_1's limitation but was incapable to forward an alternative
5. Hints given in G5 came out being inoperative; this suggests the expected conflict does no sit inside "area of immediate development". The student does not possess the required theoretical elements to with P1. Similar analysis to those above allowed to extract from the rest of the situations results such as:
6. The interpretation error about the algebraic expression of a function defined by portions is due to the C.I. stating that "**the expression of a function in portions is a simplified notation way to actually note two functions**". (**CI_1**) The student is not aware of inconsistency between **CI_1** and the function definition $[f(p)=y_1 \neq y_2=f(p)]$! Here **CI_1** appears as **an epistemological hindrance linked to the notion of function! (O_1F)**. This has prevented the emergence of the required conflict (P_1) to evolve from C_1 to C_2.

leading to believe that "**Qs 'fill' the straight line**" or that "**Is 'fill' the straight line**" is an epistemological hindrance **O_2N** which prevents "seeing" the following characteristics of real numbers **R**:

- Rational numbers, **Q**, are dense/compact within Irrational numbers, **I**.
- Irrational numbers are dense/compact within rational numbers.
- **$Q \cap I = \phi$** .
- **$Q \cup I = R$.**

The student needed first to compensate the disturbances occurred as a consequence of conceptual deficiencies such as: definition of a function in portions; definition of the notion of function, algebraic expression for f and its graph, and numerical structure of the straight line. These deficiencies were linked to the epistemological hindrances **O_1F** and **O_2N**. Both deficiencies and hindrances prevented conflict (**P_1**) between "braking" and "proximity" to appear in considering question No. 5, as would have been required to evolve conceptual scheme, hereby known as C_1, towards C_2 scheme; the student only sees the ideas to be conflicting starting from question No. 7, in as much as he has developed and balances his conceptual schemes related to the notions of function and of real number.

At the end, the student managed to achieve beta equilibrium for C_2. Evolution towards C_5 and C_6 could no be reached because of the presence of other hindrances, already highlighted by Sierpinska (1985) and Cornu (1986), and confirmed through Delgado's (1995) research, although not analyzed in this report.

6. Conclusions

This research infers **the possibility of setting up pedagogical situations which allow to keep control over the student's conceptual change**, thus performing an efficient teaching through the linkage of learning to cognitive progress.

The student manages to improve his mathematical foundations by applying the guide, and gaining awareness of conflict within the situations generated by such a tool.

The student's behavior unveiled: that he identifies the value of his own outcomes and errors, of academic discussion, of mathematical rigor, of reasoning and of mathematical prove. This identification generated a set of requirements which developed his motivation for an active involvement in mathematical debates and construction. This motivation was to be seen due to maintaining and improving an activity focused on setting out some practical relationships between the following

mathematical structures, theories and approaches:

- structure of real numbers ordered field and numerical
- structure of real straight line;

- theory of sets and theory of functions;
- mathematical logics and proving methods.

The systemic network used for establishing the defined notions the students would be likely to formulate, proved to be a very useful tool for tracing conceptual changes and in the setting out of situations for conceptual conflict. Evolution diagrams were designed to allow identifying conflicting situations and centering attention on the elements involved in them, or else, in case no conflict would arise, the reasons preventing it to do so were identified. Within those diagrams, level of balancing of conceptual schemes at the end of the process was also observed. The chart of codes eased the ranging of students' outcomes in view for a further analysis.

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Facets and Layers of the Function Concept

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*This paper considers different aspects that make up the function concept, taking critical account of several current theories of multiple representations and encapsulation of process as object to build a view of function in terms of different **facets** (representations) and different **layers** (of development via process and object). An interview technique is used to determine the profile of students according to this view.*

Facets and Layers of the Function Concept

The function concept has been a major focus of attention for the mathematics education research community over the past decade. (See Dubinsky & Harel, 1992, for example.) Schwingendorf *et al* (1992) contrast the *vertical* development of the concept in which the process aspect is encapsulated as a function concept and the *horizontal* development relating different representations. We refer to these as *depth* and *breadth* respectively (noting that increasing depth here means higher levels of cognitive abstraction) and investigate the way in which the student's concept image of function can be described in terms of these two dimensions.

The breadth dimension is often conceived as consisting of various *representations*, including *geometric*, *numeric* and *symbolic*. However, there is increasing criticism of the theory of the mental representations involved—what they actually represent, and how they are linked cognitively:

> I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of a representation... The core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.
> (Thompson, 1994, p. 39)

Crick (1994, p.10) notes the way in which the brain is built as "a messy accumulation of interacting gadgets" promoted by evolution, so that "if a new device works, no matter in however odd a manner, evolution will try to promote it". However clean and neat we attempt to formulate the mathematical theory in terms of external representations, the internal workings of the brain operate in a far more complex manner.

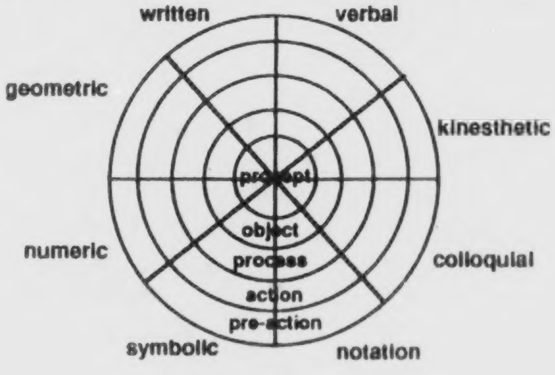
To acknowledge this debate, we use the word *facet* to build up a description of the breadth dimension. Webster's *New World Dictionary* (Guralnik, 1980 p. 300) defines a facet as "any of a number of sides or aspects." The facets of a mathematical entity refer to various ways of thinking about it and communicating to others, including verbal (spoken), written, kinesthetic (enactive), colloquial (informal or idiomatic), notational conventions, numeric, symbolic, and geometric (visual) aspects. These are not intended to be independent or exhaustive, but provide a suitably broad framework to begin an analysis of the function concept in this paper.

We use the term *layer* to refer to various levels of the depth dimension in the development via cognitive process to mental object. This has been discussed extensively in the literature, including Dubinsky's Action-Process-Object construction in which mental actions (on objects) become repeatable processes which are encapsulated as objects (see Breidenbach, *et al*, 1992; Dubinsky & Harel, 1992). In a similar way, Sfard (1992) begins with a *process* acting on familiar objects which is first *interiorized*, then *condensed* "in terms of input/output without necessarily considering its component steps" (corresponding to Dubinsky's notion of formation of a process) and then *reified* as an "object-like entity".

Gray & Tall (1994, p. 121) describe a *procept* essentially as the amalgam of three things, a *pro*cess (such as addition of three and four), a con*cept* produced by that process (the sum) and a symbol that evokes either concept or process (e.g. 3+4). Following Davis (1983), they distinguish between a process which may be carried out by a variety of different algorithms and a procedure which is a "specific algorithm for implementing a process" (p. 117). A procedure is therefore cognitively more primitive than a process.

Webster's Dictionary describes a "layer" as "a single thickness, coat, or stratum." In this paper, *action*, *process*, and *object* are considered as three layers of increasing depth. One new layer is added before action, called *pre-action*, for students at the ground floor, so to speak, with respect to a concept. After the object layer we place a *proceptual* layer, to indicate the flexibility to move easily between process and object layers as required.

The two aspects can be combined diagrammatically with the layers as concentric circles representing increasing depth, sliced into sectors representing various facets.



Facets and layers of a concept

Facets and Layers of the Function Concept

Three facets of the function concept—*numeric* using tables, *geometric* using graphs, and *symbolic* using equations—have been widely discussed in the literature (eg Cuoco, 1994; Schwingendorf *et al*, 1992; Sierpinska, 1988; Thompson, 1994). Written and verbal descriptions of function represent two other facets and the function notation is the notational facet. We will explore the colloquial facet using the notion of function machine. Finally, the kinesthetic aspect might be represented by asking students to act out their understanding about function.

Note that several of these facets have sub-facets. For example there are several ways to represent a function symbolically using symbolism such as $f(x) = x+1$ and $f: x \to x+1$. Visually, a two-dimensional coordinate graph provides a visualization for functions of one variable from the real numbers to the real numbers. Other visualizations, such as drawing correspondences from domain to range, can also be used for the geometric facet.

An area that has received much attention is students' ability to move comfortably between facets. This implies that they can choose the most appropriate facet to use for a given problem. Cuoco (1994, p. 125) suggests that the connections between "representations" are properties of a "higher-order function." While these are not the subject of this paper, it is important to appreciate the subtleties involved in linking the facets of a concept.

The layers of the function concept, especially the action-process-object layers, have received extensive treatment in the literature. According to Cuoco, "Students who view functions as actions think of a function as a sequence of isolated calculations or manipulations" (Cuoco, 1994, p. 122). Specific procedures are regarded as being at the action level. Students at this level are dependent on the procedure performed to obtain output from input. Cuoco suggests that "students who view functions as processes think of functions as dynamic (single-valued) transformations that can be composed with other transformations" (ibid, p. 122) and goes on to suggest that when students can view functions as "atomic structures that can be inputs and outputs to higher-order processes," such students have an object conception of function (ibid, p. 123). Students reach the most depth (the procept layer) when they can demonstrate flexibility in viewing a function as either a process or an object, as required by the problem situation.

Student Conceptions of Function

A number of community college students were interviewed to begin to analyze their concept image of function in terms of facets and layers. In this paper we report interviews with one student who had just completed a "reform" developmental algebra course. The text (DeMarois, McGowen & Whitkanack, 1996) focuses on student investigation of problems. This is based on a pedagogical approach that uses a constructivist theoretical perspective of how mathematics is learned (Davis *et al*, 1990). The authors subscribe to the theoretical perspective that the main concern in mathematics should be "with the students' construction of schemas for understanding concepts. Instruction should be dedicated to inducing students to make these constructions and helping them along in the process." (Dubinsky,1991, p. 119). Each unit begins with an investigation of a problem situation. Following the gathering of data, students work collaboratively on tasks based on the investigation activities. A discussion in the text summarizes essential mathematical ideas. The instructor orchestrates inter-group and class discussions of the investigations. Explorations are assigned to reinforce the knowledge students are expected to have constructed during successive steps of the cycle.

The materials focus on development of mathematical ideas using a core concept of function. Each function is based in a problem situation. Functions are often investigated numerically, graphically, and with function machines before the symbolic form is created. As Sierpinska writes: "The most fundamental conception of function is that of a relationship between variable magnitudes. If this is not developed, representations such as equations and graphs lose their meaning and become isolated from one another." (Sierpinska, p. 572) As each new function arises, investigations support multiple facets and wise choices in terms of what facet might be best for analyzing a specific problem. Tables, equations, graphs, function machines, verbal and written descriptions are all used to analyze relationships. Graphing calculators provide excellent support for the tables, equations, and graphs.

We summarise the interviews with DK who had completed two semesters of developmental algebra, receiving an "A" in both semesters.

Layers of the Verbal Definition Facet

The first question explores the student's verbal definition of function.

Int Explain in a sentence or so what you think a function is. If you can give a definition for a function then do so.

DK A function comprises, I think, the whole general idea of what we have been doing. And some functions you go into relationships, from there you go into equations, models, quadratic or linear. I mean, everything comes off of the function. I think that's a basic idea in mathematics, the function.

Int Let's narrow it down a bit more.

DK If I saw an equation, I would call that a function.

Int Anything else?

DK A relationship. To me a function is the whole general idea.

DK tends to be very non-specific about function. It seems that she has spent so much time studying problems that relate to functions that she has overgeneralized. When asked to be more specific, she says she would call an equation a function, but she places no restrictions on equation. Finally, she uses a key description that the materials emphasize: relationship. However, she places no conditions on the relationship. Ultimately her verbal definition of function shows no depth suggesting that she is at the "pre-action" layer with respect to the verbal definition facet.

Layers of the Notation Facet

Another crucial part of working with functions is understanding function notation. DK expressed some confusion about when to read a string such as $y(x)$ as multiplication and when to read it as function notation:

Int What do you think when you see the notation $y(x)$?

DK That's a function notation. x means—x is the input. y is the output. When you substitute a number in with the x, then you would, on the other side of the equals sign, apply that number to all the xs in that equation.

While DK has interpreted the notation correctly, she seems very procedural in her use of it. She finds it difficult to accept $y(x)$ alone without setting it equal to an algebraic expression that describes the process. Thompson (1994, p. 24) suggests that "the predominant image evoked in a student by the word "function" is of two written expressions separated by an equal sign." DK seems to have this image. We might interpret that she is at an action layer with respect to the notation facet.

Using Function Composition to Probe the Depth of Numeric, Geometric and Symbolic Facets

None of the students had been exposed to composition of functions before. The interviewer provided some brief comments on the meaning of function composition prior to the following questions. In addition to gaining information about students' ability to answer questions about the numeric, geometric, and symbolic facets, these questions permitted more analysis of the students' understanding of the notation facet.

The students were given two input-output tables, one for function f and the other for function g.

x	$f(x)$
1	3
2	–1
3	1
4	0
5	–2

x	$g(x)$
–2	3
–1	1
0	5
1	2
2	4

Int What is the value of $f(g(2))$? Why?

DK exhibits much confusion.

Int Let's break it down a bit. Can you tell me what g of 2 is?

DK g of 2 is 1. Input of 1. g of 2 is the x of 1.

Int Is 2 a value for input or output in this case?

DK 2 of x is the input. I better put a 2 over here (she points to the input column).

Int So g of 2 is equal to what?

DK 4.

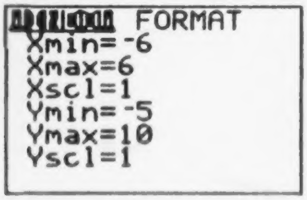
Int So why don't you substitute 4 for $g(2)$ is the expression $f(g(2))$?

DK f of 4 is zero. It equals zero. f of g of 2.

This was the correct answer. DK tried to analyze what she had done, but became hopelessly confused between input and output.

The interviewer continued to question DK on composition, this time focussing on the geometric facet. She initially demonstrated that she could interpret expressions of the form f(a) and g(b) where a and b are given from the graph.

Int Consider the following graphs for functions *f* and *g*. The graph of *f* is the line. The graph of *g* is the parabola.



```
WINDOW FORMAT
Xmin=-6
Xmax=6
Xscl=1
Ymin=-5
Ymax=10
Yscl=1
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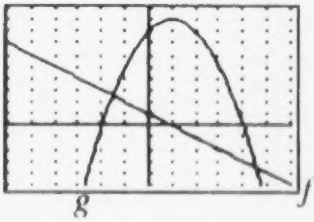


Approximate the value of $g(f(2))$. Describe what you did.

DK was unable to even begin to answer this question, revealing a weakness to deal with the geometric facet.

Composition was then investigated in the symbolic facet:

Int Consider functions *f* and *g* defined as $f(x) = 3x - 5$ and $g(x) = x^2 + 1$. What is $g(f(3))$? Describe what you did.

DK Well, *f* of 3 equals *x* squared plus...

Int What did you do?

DK Maybe I am confused.

Int Suppose I covered this (the *g* of) up.

DK Then I would use *f* of 3.

Int And what would you get?

DK 4.

Int So this could be interpreted as *g* of what?

DK *g* of 4.

Int And what would that mean?

DK Oh, okay. *g* of 4.

DK writes 17, revealing that, though she has difficulty dealing with the numeric aspect of composition of functions using tables, she is perfectly capable of interpreting the symbolism numerically. This was confirmed by another example:

Int Suppose I want *g* of *f* of 1.

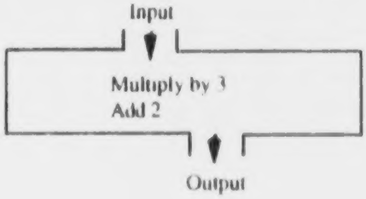
DK Well, I'd do this first. I'd go *f* of 1 equals 3 *x* minus 5. f of 1 equals 3 minus 5. *f* of 1 equals -2. You want *g* of *f* of 1? And then you take that and *g* of negative 2 equals *x* squared plus 1. Positive 4 plus 1 equals 5. Is that right?

DK was more able to deal with function composition symbolically than in the other two representations. This may partly be due to the fact that she had dealt with the concept in two previous problems. She still, however, initially needed help interpreting the notation. She appears very adept (procedural?) at finding an output given an input symbolically. She appears very comfortable with pushing the symbols and performs satisfactorily with the action level of the symbolic facet yet struggles with the action facet of the numeric and graphic.

Layers of the Colloquial (Function Machine) Facet

The materials use function machines extensively to analyze functions. This is the first facet of function, after a written, informal definition, that the students interact with.

Int Consider the following function machine.



What is the output if the input is $y(x) = x^2 - 5x$? What did you do?

DK What is the output if the input is–okay. If the input is x squared minus 5 x. You'd multiply it by 3 and then you'd add 2. Do you want me to work it out?

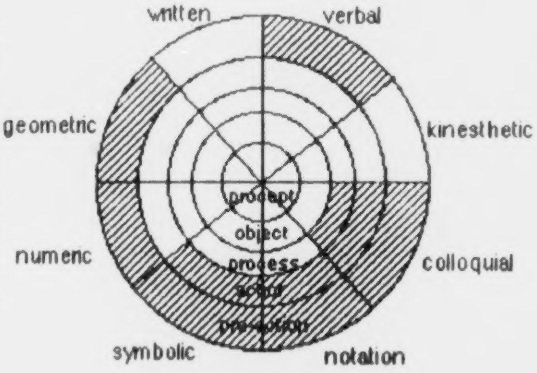
Int Sure

DK 3 x squared minus 15 x plus 2.

The symbolic input caused little trouble. On the colloquial facet, she continues to be able to handle a symbolic as well as a numerical input, suggesting she is moving towards the process layer.

Analysis

Based on the responses to these few questions, we begin to develop a profile of the student's understanding of function. The shading indicates the number of layers the student has demonstrated in their understanding of a specific facet. The student's knowledge of a specific facet has not been assessed if the outermost layer (pre-action) is unshaded, in this case the written and kinesthetic facets.



DK's profile

Reflections

These interviews underline the complexity of the function concept, for instance that the student concerned can operate more successfully in the symbolic facet than in the numeric facet, even though symbolism seems to be more sophisticated than numeric representations. Further work is necessary to complete the student's profile and critical issues have arisen in the classification of facets, including the links between them and deeper analysis of sub-facets. Nevertheless, the profile provides useful insight into a highly complex issue, re-focusing our attention on the nature of the cognitive structure of the function concept.

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EVOLUTION OF CONCEPTIONS IN A MATHEMATICAL MODELLING EDUCATIONAL CONTEXT

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Abstract

We deepen the problem of conception evolution, focusing our attention on the conceptions that more frequently come out in a long term experimentation on the sun shadow phenomenon. This analysis points out that naive conceptions, expressed by the pupils before a systematic study of the phenomenon is performed, usually evolve due to the classroom activity, even when they are not explicitly object of didactic intervention, either interlacing with new conceptions, or being substituted by them. Using one or another conception seems to depend on various factors of cognitive relevance. The result is a not linear personal path, where naive conceptions can come out again even after more scientific ones have been learned and successfully applied.

1 - Introduction

In a previous paper, Boero et al. (1995a) discussed some components of a mathematical modelisation of the sun shadow phenomenon. That experience pointed out that, despite a good school knowledge of the mathematical components, naive pre-geometrical conceptions may come out again in particularly complex problem situations.

In this paper, we deepen the problem of conception evolution in the long period. We focus our attention on the conceptions, both naive and mediated by the teacher, that most frequently come out in a long term experimentation (one and a half school year) in an intermediate school where the sun shadow phenomenon is object of a systematic study aiming at constructing a mathematical model. By conception, we mean anything a person thinks about the working of a phenomenon, independently of the origin of this thought, from unstructured experiences up to the conscious learning of a model. In fact, it is mostly impossible to distinguish the origin of some conception expressed by a pupil simply from an analysis of his works. We chose this field of experience for this analysis of the evolution of conceptions since it regards a phenomenon whose existence is known to everybody, so that it can stimulate and help growing a body of shared knowledge. Moreover, it is extremely rich from a mathematical point of view.

The results of our analysis can be summarised as follows:

- in the first approach to the study of the sun shadow phenomenon, different conceptions, both non-geometric or pre-geometric, are expressed by different pupils in a same task, or by a same pupil in different tasks;
- due to classroom activities, generally these conceptions evolve, even though they are not explicitly target of didactical intervention; some of them become interlaced with elements of the mathematical model introduced by the teacher, others disappear;

- for every student, using one or another conception seems to depend on various factors, such as the characteristics of the assignment and the didactic itinerary. The result is a not linear personal path, where good command of the mathematical model can be followed by performances where the student goes back to naive conceptions of the phenomenon. A same conception can show up in successive moments at different levels of maturity, not always increasing.

2 - Experimental design

We analysed the individual written works (texts and drawings) of 34 pupils of two classes (that we will call Group A), taught from grade VI to grade VII by the same teacher. These works regard a sequence of 5 assignments, and result particularly rich as concerns the number and variety of conceptions, both of scholastic origin or not. These assignments belong to a didactic itinerary of about 80 hours of classroom activity on the sun shadow phenomenon, intended as *field of experience* (Boero et al, 1995b). For the sake of comparison, we took into consideration also the written works, only concerning the last assignment, produced by:
- the pupils of 4 classes in grades VI, VII, VIII (from the same school, that we will call Group B), who did not analyse in school the sun shadow phenomenon;
- the pupils of 4 classes in grade VII (from another school, that we will call Goup C) who had followed an analogous didactic itinerary on the sun shadow problem, with the same duration, but giving a different relevance to the various classroom activities.

The first assignment is a questionnaire, proposed at the beginning of grade VI (September 94): *Did you notice that in sunny days your body makes a shadow on the ground? Is your shadow longer at 9 am or 11 am? Why?*

The second assignment (end of December 94) was a composition: *What is for me the shadow.*

The third one, a request for a reasoned discussion, *Try to explain how you think that shadows are formed; aid yourself with a drawing and try to explain also the different lengths of shadows*, was assigned few days later (beginning of January), without any intermediate activity on this topic.

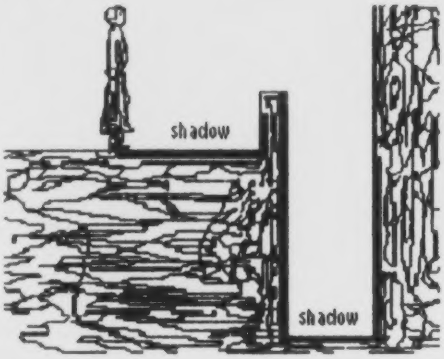
At this point, a systematic study of the shadow phenomenon was started and conducted until next October. It was developed by means of both individual activities and common discussions. Individual activities concerned problem resolutions and formulation of previsional and interpretative hypotheses, always requesting the pupils to verbalise their reasoning. The common discussions aimed at comparing hypotheses, planning verifications, reaching a shared synthesis of the learned facts.
This phase included the following activities:
- comparison and discussion of the different conceptions come out from the initial assignments, yet without taking a strong position on the validity of each of them;
- observation and description of the phenomenon in different moments of the same day, and in different days;
- observation and description of the parallelism of the shadows of vertical poles, and of the parallelism of the sun rays;
- study of the rotation angle of shadows;
- computation of the height of objects that can not be directly measured, based on the measure of their shadows;

- geometrical elaboration of the performed observations up to Talete's theorem;
- definition of the "shadow triangle" as a model for the phenomenon, including the definition and evaluation of the angular height of the sun.

About two months after the end of this study phase, a fourth assignment was given, consisting in a request for an interpretation and reasoned discussion: "It is a summer day, there is a beautiful sun and a white wall. At a certain point, you see your own shadow going up on the wall. How can you thoroughly explain the phenomenon of the climbing shadow? If you want, aid yourself with some drawings".

The fifth assignment, proposed few days later without any other intermediate classroom activity, consists in a problem resolution:

The drawing at the right represents in section a situation of shadows made by the sun. A woman is coming close to the low wall and is represented with her shadow. On the other side of the low wall is a deep hollow space and then another high wall. Draw where will be the shadow of the woman in the picture when she moves three steps forward.



In some classes the shadow in the hollow space was not marked. This fact, however, did not seem to influence the obtained results.

3 - An analysis of the pupil's written works

The following 8 conceptions come out most of the times from an analysis of the pupil's written works:

C0 - "Strength-length": a shadow's length depends on the strength and brightness of the sun. This conception is detected through explicit declarations of the pupils.

C1 - "Shadow as duplicate": a shadow is a specular projection of the subject producing it. This conception is detected mostly in the drawings.

C2 - "Shadow as attachment": a shadow is an attachment of the body producing it; it is attached to the body and is dragged according to its movements. This conception is detected in the drawings or in explicit declarations.

C3 - "Pre-geometrical": a shadow is a projection of an object, at the opposite side of the sun with respect to the object producing it. This conception expresses a set of spatial relations between shadow and sun, such as: low sun, long shadow; it is detected in the drawings or explicit declarations.

C4 - "Metric regularity": a shadow preserves its length under some conditions; objects with equal height have shadows of equal height. This conception is detected in explicit declarations and in drawings realised with exact measurements.

C5 - "Shadow triangle": the length and direction of a shadow, the heigth of the object producing it and the direction of the sun rays are related so that any two of them can determine the third one. This conception is detected mostly in the drawings and occasionally in verbal declarations.

C6 - "Shadow space": a shadow is the visible effect of lack of light determined by some object hit by the sun. This conception is detected mostly in the drawings and occasionally through verbal declarations.

C7 - "Ray parallelism": the metric regularity of shadows depends on the fact that the sun rays are parallel and do not come out of the sun in many directions, as popular culture suggests. This conception is detected in the drawings.

The first four conceptions can be considered spontaneous, since we frequently found them also in students of Group C. They have been detected also by previous studies (Boero et al 1995a). They take into account and interpret, each one in its own way, several perceptive aspects of the sun shadow phenomenon: for C0, the neatness of shadows in the middle of the day, which gives a clearer perception of them; for C1, the fact that, when the sun is low, a shadow on a wall is very resemblant to the person or object that originates it; for C2 the fact that our shadow really seems an attachment that we carry with us, which, starting from our feet, lies on and adheres to the surfaces where it is projected; for C3, different perceptive evidences, like those concerning the opposite position of the shadow with respect to the sun, or the fact that shadows are longer at sunset., when the sun is clearly low.

As concerns conceptions C4, C5, C6, C7 we can hypothesise that they are a product of the work on the mathematical modelization of the phenomenon. Indeed, as also previous experiences have pointed out (Boero et al. 95b), these conceptions do not spontaneously come out in most pupils, not even when solving assignments that seem to lead toward these conceptions.

The performed analysis of the pupil's written works is summarised in table 1.

4 - Considerations on the performed analysis

4.1 - The assignment influences the (more or less unconscious) choice of a conception

Every assignment makes reference to one or more aspects of the sun shadow phenomenon. Hence, it is clear that it can recall some conceptions more than other ones. For instance, in the initial questionnaire, the comparison between shadow lengths at 9 am and at 11 am frequently recalls the conception of the shadow length depending on the sun brightness, and the pre-geometric one. On the other hand, in Assignments 2 and 3, not only different conceptions come out, but they are used in different percentages in the two assignments, despite there was no didactical intervention between them.

4.2 - The didactic itinerary influences the choice of a conception

As it was expectable, big differences can be detected, in the last assignment, between the classes that did follow an itinerary on the sun shadows and classes that did not, especially concerning the presence of conceptions C4, C5, C6, C7. This fact confirms that these four conceptions are not spontaneous but produced by didactical mediation. Moreover, the classes from Group C show in this assignment a different distribution of conceptions. In particular, in these classes, several pupils use different conceptions in a same resolution (in particular C1 and C5), without noticing that they are in conflict with each other. Some contradictions are present also in the classes from Group A but less frequently and concern the coexistence of closer conceptions; compare, for example, Figure 3 (from Group A) and Figure 4 (from Group C).

If we try to point out the differences between the didactic itineraries carried on in Group A and in Group C, the following points must be noticed:

- kind of verbalisation requested from the pupils: in Group C it has mainly the function of reporting on the problem solving procedures, while in Group A its function is mainly the construction and justification of pupil's reasoning, which can have lead the students to notice the possible contradictions in their works;
- discussions on the first assignments, aiming to letting pupils become aware of the variety of their original conceptions; this phase of discussion appears important to make pupils ready to start an evolution process of their conceptions;
- modality of introduction of the mathematical model of the shadow triangle: in School B, this model was introduced rather early, and applied in most problem resolutions. In School A, the formal introduction took place towards the end of the itinerary. Because of its late introduction, this model was not the only resolution tool for School A pupils, who had several chances to experience other resolution tools; as a consequence, pupils tended to use it when they felt it was needed, rather than automatically in any situation.

4.3 - The didactic itinerary influences the conception evolution

Due to the classroom activities, the non geometrical and pre-geometrical conceptions evolve and coexist with elements of the geometrical ones. For example, we find that "preserving the shadow length" can coexist with the "shadow attachment" (see fig. 1) or "shadow duplicate", in particular in Assignment 5; the "shadow" triangle" coexists with the "metric regularity" (see fig. 2), in particular in Assignment 4. Finally, we can individuate some typical conception evolution path among pupils from School A: C0 soon disappears, and, it seems, irreversibly; the pre-geometric C3, in percentage rather common at the beginning, though in different proportions in the various assignments, disappears at the end of the experience, leaving way again to the "shadow as attachment" or to the acquisition of the geometrical conceptions. The "shadow as attachment" is progressively integrated with the "metric regularity" and with the "shadow triangle", but this integration fails in Assignment 5, where we found it conflicting with the "length preservation" (see fig. 3).

Geometrical conceptions mediated through the didactic itinerary gradually seem to get acquired, and usually do not coexist with naive conceptions. However, all pupils who don't use C3 in Assignment 3, are not able to use C5, C6 and C7 in the subsequent assignments. Hence, the acquisition of pre-geometrical conceptions seem a crucial point for evolving toward strong geometrical conceptions. Moreover, we think that the "shadow triangle" is not used by about one half of the pupils in Assignment 5 probably because it was mostly presented as a functional dependence of the shadow length on the sun direction and object height, rather than as a relation of these three elements, which makes it difficult to apply when the functional dependence needs to be reversed.

4.4 The used representation system influences the evolution of conceptions

The semiotic system used in the phase of study carried on in the classroom creates a working environment where it is possible to become aware of the conflicts between different conceptions and overcome them. Hence it can lead to an understanding, internalisation and evolution of conceptions, as well as to a better comprehension both of the phenomenon and of its modelization, as it is discussed in (Mariotti 96, Scali 94).

5 - Concluding Remarks

The analysed set of conceptions probably does not include all possible conceptions about the sun shadow phenomenon; other assignments might have let come out other ones. However, this does not limit the validity of our analysis, since our aim is not to study the conceptions about a particular phenomenon, but to verify if and how the conceptions about a phenomenon which is an important field of experience can evolve due to a didactical intervention. This is a key issue when studying children learning processes.

A delicate issue concerns the dependence of the performed analysis from the didactic itinerary. In this respect, it is necessary to distinguish general results, such as the plurality of conceptions used by a single pupil and the integration of naive conceptions with other mediated by the teacher, from particular results, such as the percentage distribution of conceptions in different assignments and typical long period behaviours. General results don't seem to depend on the particular didactic itinerary, though we base this conjecture only on two groups of classes with partially different didactic itineraries. On the other hand, particular aspects appear to be influenced by the didactic itinerary carried on. In particular, we see this in Assignment 5: in the classes from Group C we can notice an extensive presence of the "shadow as duplicate" together with the conflicting "shadow triangle". In Groups A abd B the "ray parallelism" is almost equally frequent in Assignment 5, while in Group C it is almost absent.

The results of this investigation appear interesting under several respects. In a teacher perspective, they stimulate teachers to a greater attention to the variety of products that pupils can realise under particular conditions, and to the resources to which they can relay on. As concerns educational research, these results pose the problem of which didactic mediations is more effective to guide the interplay between the natural resources of the pupils and those offered by the teacher. As concerns both educational and psychological researches, they point out how crucial is the choice of representation systems and graphical models in order to favour a conception evolution, and pose the problem of choosing the right way and time to introduce the knowledge and use of adequate semiotic systems and fruitfully reflect on them.

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Pupil	assignment 1	assignment 2	assignment 3	assignment 4	assignment 5
Class A					
1	C0	C1	C3	C1-C2	C4
2	C3	C1	C1-C2-C3	C3	C2
3	C3	C2	C1-C2	C5-C6	C2
4		C1	C3	C1-C2	C4
5	C3	C2	C3	C3	C4
7		C1	C3	C1-C6	C5
9		C1	C2	C1	C1-C2
10	C3	C1-C2-C3	C3	C4	C1-C2
11		C1-C2	C3	C4-C6	(C5)-C4
12	C0	C1	C3	C5-C6	C4
13	C3		C3-C6	C5-C6-C7	C5-C7
14	C3	C3	C3	C5-C6	C5-C6-C7
15		C1	C3	C4-C5	
16	C0		C3		C5
17	C0	C2	C3	C5-C6-C7	C2
Class B					
1	C3	C2	C3	C2	C4
4	C0	C2	C3	C5	(C5)-C4
5		C1			C5-C7
6					(C5)-C2
7	C0	C3	C3	C5	C5-C5-C7
8	C3	C1	C5		C1-C4
10		C3	C3	C5-C6	C4-C5-C6-C7
11		C2	C3		C5
12	C3	C2	C2		C4
13	C3	C1-C2	C3	ambiguous	C4-C5-C6-C7
14	C3	C3			C2-C4
15	C0	C1	C2	C1-C4	C2-C4
16		C1		C4	C4
17		C2	C3	C4	C4
18		C3		C5	C5-C7
19		C2	C3	C4	C4
20		C3	C3	C5	(C5)-C4
21	C0	C2	C3	C4-C5	C5-C7
22	C3	C1		C5	C2

Table 1: Conceptions detected in the pupil's written works

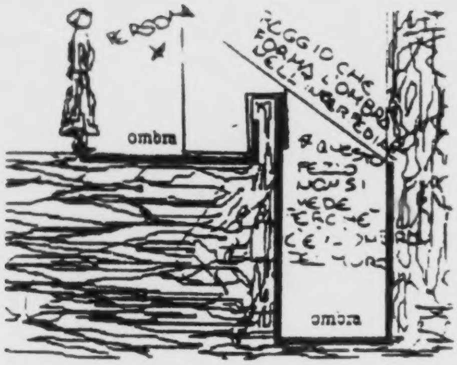
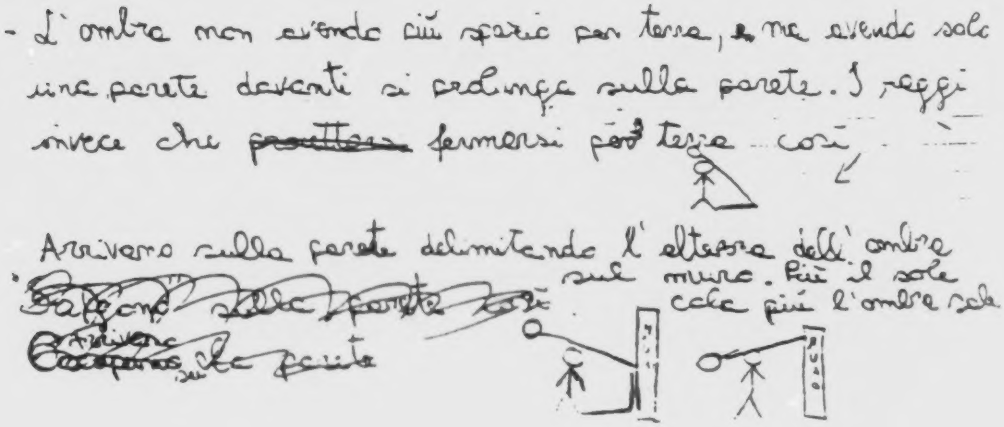

PERSONA
RAGGIO CHE FORMA L'OMBRA DELL'INTERMEDIARIO
ombra
QUESTO PEZZO NON SI VEDE PERCHÉ C'È IL MURO
ombra

Figure 1

The text in Fig. 2 says: "The shadow, not having more room on the ground, but having only a wall in front, extends on the wall. The rays, instead of ~~projecting~~ stopping on the ground like that ...
Arrive on the wall limiting the heigth of the shadow on the wall. The more the sun gets down, the more the shadow gets up".



- L'ombra non avendo più spazio per terra, ~~e~~ ma avendo solo una parete davanti si prolunga sulla parete. I raggi invece che ~~proiettarsi~~ fermarsi per terra così

Arrivero sulla parete delimitando l'altezza dell'ombra sul muro. Più il sole cala più l'ombra sale

~~Battono sulla parete così~~
~~Arrivano~~ ~~Occupano~~ su la parete

Figure 2

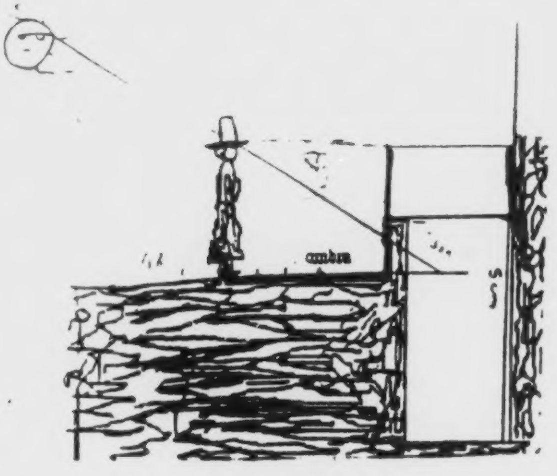


Figure 3

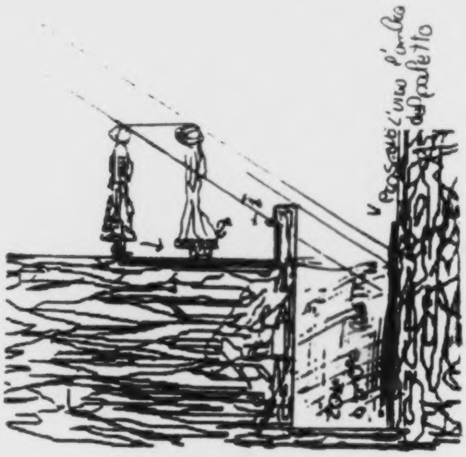

Poserà sul muro l'ombra del paletto

Figure 4

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THE ROLE OF PHYSICAL PHENOMENA IN STUDENTS' INVESTIGATIONS OF THE TANGENT FUNCTION

Helen M. Doerr

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Abstract

The analysis of four teaching interviews yielded several insights into the students' construction of the tangent function through an investigation of the magnitude of the horizontal force required to hold an object suspended on a rope at a given angle. The students moved from the physical apparatus to both qualitative and quantitative expressions about the relationship between the angle and the force. The students frequently focused on boundary conditions as sense-making points and confirming instances for their proposed relationships. The students demonstrated an openness to several possible conjectures while in the process of validating a meaningful relationship. This research also sheds some light on students' understanding of the nature of the vertical asymptote of the tangent function.

Introduction

The difficulties encountered by students in constructing equations when given the data or graph are well known and yet only limited research has been conducted in this area (Leinhardt, Zaslavsky, & Stein, 1990). Even more limited is the research has been done on student's understandings of trigonometric functions; see, for example, Doerr and Confrey (1994). This study was designed to investigate student's construction of the tangent function in an open-ended problem situation involving the force required to hold an object suspended on a rope at varying angles of displacement.

The theoretical framework for this study builds on Confrey's (1993) notion of concept development as arising from the interplay between grounded activity and systematic inquiry. Students' interactions with the physical apparatus provide powerful beginning points for their inquiry into the mathematical relationship between two covarying quantities: angle and force. The apparatus serves as a tool to guide the inquiry and to confirm the students' conjectures about possible relationships. The context of the problem is a source of experimentation and of data. However, the context by itself is not sufficient. The students need to engage with the problem in the authentic sense argued for by van Reeuwijk (1995) and others in their program of realistic maths education: "The contexts should make sense to the students, and they must support the students in solving the problems and developing (or reinventing) mathematics" (p. 136). In engaging with the problem situation and the evolving mathematics, students have found supporting computer technology for representing tables and graphs to be powerful tools for investigating functional relationships (Doerr, 1994).

In this study, the interplay of the physical experimentation with tables, graphs and equations, and the role of the geometric representation (right triangle diagrams) are analyzed. The task posed to the students is to construct a meaningful relationship beginning with the data collected in a physical experiment. The validation as well as the explanatory and descriptive power of the equation are critical issues to be examined as the students engage with the problem. In addition, this research sheds some light on students' understanding of the nature of the vertical asymptote of the tangent function, both graphically and in terms of the physical phenomena of a pulling force.

Methodology, Data Sources and Analysis

This paper presents the results of the student investigations of the tangent function which occurred after the second sub-unit, the effect of multiple vectors (e.g. forces) acting on an object, in an integrated mathematics and science curriculum. The curriculum was designed as part of a larger research project on a modeling approach for building student understanding of the concept of force and enhancing problem-solving skills. The setting for this study was an alternative community school, part of a small urban public school system. The study took place in an integrated algebra, trigonometry, and physics class with 17 students in grades 9 through 12, who had elected to take the course. Four students representing a range of achievement were selected for individual teaching interviews. The interviews were conducted after the related material had been covered in class. All interviews were conducted within a three-day period. Each interview was video-taped and transcribed for analysis. The students' computer work and paper and pencil drawings were also collected for inclusion in the analysis.

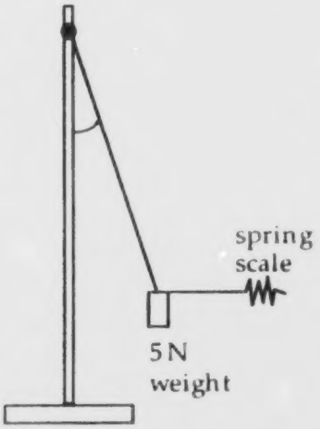
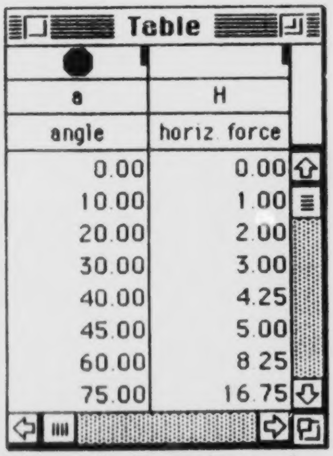


Figure 1. Physical Apparatus for the Windy Day Problem

The interview problem was designed to provide an opportunity to better understand the student's content knowledge and to observe the students' spontaneous use of and choices among the available representations and tools. The problem required the students to develop and use relationships that described or explained a physical event. The following open-ended problem situation for exploring the tangent function and the effect of multiple vectors acting on a point was presented to each of the four students:

> One day, Sandy and her friends decided to analyze the force of the wind on an object hanging on a rope. They observed that the object hung straight down when no wind was blowing. When there was just a slight breeze, the object moved to a position at a slight angle from the vertical. When the wind was

> stronger, the object moved further out and the angle was bigger. When the wind was very, very strong, like in a hurricane, the object moved to a position that was nearly horizontal.
>
> What is the relationship between the angle from the vertical and the force of the wind for an object of a given weight?



a	H
angle	horiz force
0.00	0.00
10.00	1.00
20.00	2.00
30.00	3.00
40.00	4.25
45.00	5.00
60.00	8.25
75.00	16.75

Figure 2. Experimental Data

The students had the physical apparatus shown in Figure 1 available to them as well as the multi-representational, analytic tool, Function Probe (Confrey, 1992). Each student began by collecting the measurement of the horizontal force using the spring scale for various settings of the vertical angle. This data was then entered into a Function Probe table as shown in Figure 2.

Results

The analysis of the teaching interviews yielded several insights into the students' use of the available tools, data, and representations. First, the students moved from the physical apparatus to both qualitative and quantitative expressions about the event. Second, the students frequently focused on boundary conditions as sense-making points and confirming instances for both their qualitative arguments and quantitative relationships. Third, the students demonstrated an openness to several possible conjectures while in the process of validating a meaningful relationship. Each of these results, with the supporting evidence from the interview data, is discussed in turn.

The students moved from their observations and their data from the physical apparatus to qualitative and quantitative expressions about the event. Alycia began her analysis by confirming the observation that the stronger the wind, the bigger the angle. After gathering her data, Aria moved from the physical sense of the pull on the rope to the behavior of the tangent function at its asymptote:

AR: But an 90, it seems like, it would be like one of those, the tangent graphs. The way it sort of approaches a line at 90, but never quite reaches it.

H: Hmm hmm

AR: I mean that's what it looks like it's going to do.

H: So what would happen as you were in here. To these values

AR: Well, then it would keep on getting higher and higher and it would be up in the millions and billions and trillions and just get way up there. Like trying to pull it, so that it's straight. You just have to keep pulling and

you make this much progress (indicates small amount with fingers) and you're pulling about 10 more newtons and then
H: But you're only making a little bit of progress in terms of getting it
AR: And then you make even less and less, the more you pull

Aria's interactions with the physical apparatus led to both qualitative and quantitative expressions of the functional relationship between the angle of displacement and the horizontal force.

The students frequently confirmed their hypotheses or intuitions about a problem by examining the boundary conditions. All of the students used the observation that when the angle is zero the horizontal force is zero. More significantly, all of the students recognized that when the angle is 45 degrees, then the horizontal force is equal to the weight of the object and that this latter event probably will provide an important insight into the problem. Sally's approach is particularly illuminating in this regard. She began by deciding to collect the data point at 45 degrees and she immediately recognized that at 45 degrees the horizontal force is exactly equal to the weight of the object:

S: OK. Now I guess I'll do, if I can get it at 45 that would be good.
...
H: And what do I get?
S: Five newtons. Oh, that's handy that when it's, that's exactly the weight of this.

The interviewer then asked her *why* she suspected that this might be useful. Sally turned to her diagram of the forces acting on the object (see Figure 3) and suggested that maybe the vertical and horizontal forces have to be equal, but she was clearly not sure. The interviewer suggested that Sally think about vertical force (the weight) and Sally recognized that the force in the rope must be balancing the weight of the object, although it was not clear that she saw that the weight must equal the vertical component of the rope force:

H: Can we think of a force vertical? Where are the forces that acting in here? Here's the force, there's that horizontal force.
S: yeah
H: That turned out to be five, five newtons
S: Um, I'm thinking this is pulling down five newtons.
H: Right. And how do, you know that because?
S: that's the weight of the object
H: So if there's five newtons going down, there's another five newtons?
S: Going up
H: Yeah.

S: Well, this [the rope] is pulling

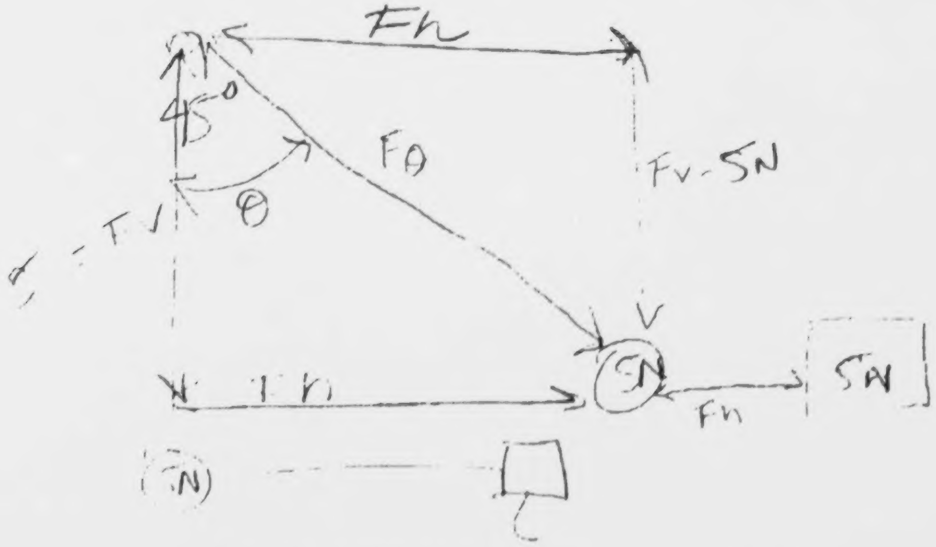


Figure 3. Sally's Force Diagram

Sally was still not clear *why* the vertical and horizontal forces must be equal at 45 degrees. The interviewer suggested that they return to the data gathering that they were doing, but Sally wanted to hold on to this event at 45 degrees a little bit longer. It was clear that she recognized this as a significant event. She then reasoned that all the force would have to be horizontal at 90 degrees; later in the interview she made it clear that it is impossible to actually exert sufficient force at 90 degrees. She knows that at zero degrees, the horizontal force is zero. She held onto her strong intuitive sense that the horizontal force equals the weight when the rope is at 45 degrees, but the interviewer moved Sally on to the data collection:

S: Uh. I still don't really see why
H: OK
S: Um
H: All right. Let's get some more data.
S: OK
H: Right. I mean
S: I just can't quite visualize cause um. When we were doing it before, it was very clear to me that when it was at 90 degrees, like the um horizontal would be um carrying it all.
H: Uh huh
S: And then when it was straight down the horizontal wasn't pulling

Later in the interview, Sally graphed the data she had collected for the horizontal force and the angle. She then fit this data with a tangent curve and confirmed that a value of 5 newtons at the 45 degree angle made sense. Sally did not go back to her force diagram to confirm her equation with the right triangle trigonometry she had started with.

The interviews confirmed that the students had made sense of the horizontal and vertical components of a force and their understanding that these components must balance (or have a vector sum of zero) when an object is motionless. Aria moved between the forces acting in the physical apparatus at 45 degrees and the geometry of the force diagram, as she argued that the force in the rope must be equal to the vector sum of the force pulling down (the weight) and the force pulling out (the horizontal force of the wind). Alycia also recognized that the forces must have a vector sum of zero when the object is not moving and saw that in terms of the balance between the horizontal forces and the vertical forces. Unlike Aria, though, she puzzled over the balance of the horizontal forces. She still had some uncertainty about the horizontal component of the force of the rope:

AL: so it's the sideways pull times 5 because um it's not moving.
H: Cause it's not moving.
AL: So, this, this horizontal force, this is equal to that.
H: Those two horizontal forces are equal.
AL: Hm, hm
H: And the two, and
AL: That's weird even though there's no rope pulling in the opposite direction,
H: right
AL: it's, even though there's no rope here, it, it still has the equal force

This interview was conducted just after the students had completed their first investigations of the resolution of a force vector into its components. As the overall unit progressed, Alycia appeared much more certain and skillful in her reasoning and use of the notion of the components of a force vector. Using the physical apparatus to find the equality of the vertical and horizontal components at 45 degrees was a strong sense-making and grounding experience for the students.

The students demonstrated an openness to possible conjectures about which prototypical function might be relevant to their data set. All four of the students posited initial conjectures about the relationship of the horizontal force to the angle; an inverse (hyperbolic) relationship, an exponential and a parabolic relationship were initially conjectured. One student, Mark, showed considerable willingness to pursue his initial conjecture. After he had collected and graphed his data, his first conjecture was that the relationship between the angle of displacement from the vertical and the horizontal force of the wind might be parabolic. Mark then attempted to fit a parabolic curve to his data through a horizontal stretch using Function Probe (see

Figure 4). The horizontal stretch of the parabola didn't quite match his data, so he attempted to fit it with a vertical stretch, which also failed. Then Mark suspected that an increase in the amount of the stretch might help to fit his data. He persisted confidently in following this stretching strategy:

> M: I may have stretched it the wrong way. (pause) Let's try the other way. (pause)
>
> ...
>
> M: Well, this doesn't seem to be working.
> H: OK
> M: Unless I have to stretch it by a lot more than this. Which is certainly possible, if not likely.
> H: OK
> M: But I'm going to follow this wild goose chase to the bitter end. (unintell) explain this force. (pause) (unintell) give up on it and come back to it in an hour and say oh.

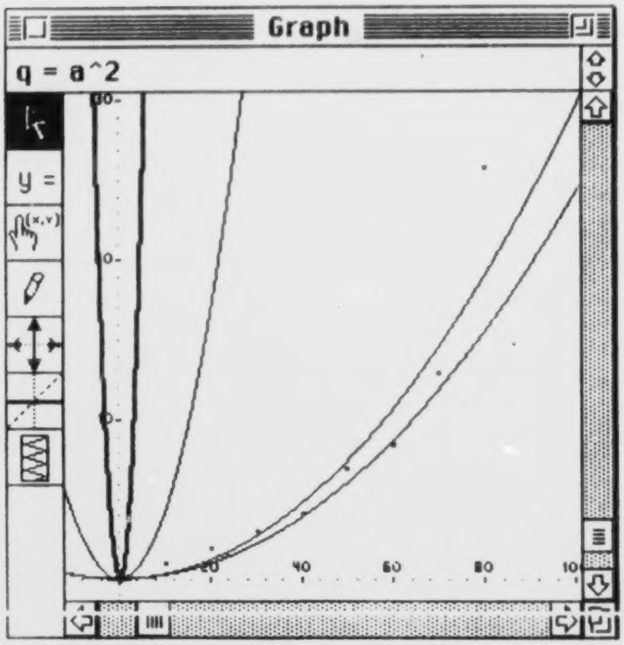


Figure 4. Stretching a Parabola to Fit the Data

Finally, Mark concluded that no matter how much he stretched the parabolic curve, it did not fit his data. He recognized that the steepness of his data near 90 degrees was the source of the problem. He concluded that the parabolic curve didn't seem to be the right one. At this point, the interviewer reminded Mark of the trigonometric curves and he quickly fitted a tangent curve to his data. Later in the interview, he confirmed this empirical relationship with an argument from a force diagram. His exploration of the alternative conjecture was marked by confidence and persistence. Ultimately, Mark coordinated the representation of his data table, with the graph, with a geometric representation and with an equation.

Conclusions

The teaching interview data shows that the students moved flexibly among the physical apparatus, the data table, the graphs of the data, the geometry of the force diagrams, and the equations that represent both the graphs and the trigonometric relationships from the geometry of the force diagram. The physical apparatus provided a grounding experience from which the students posited qualitative *arguments, possible conjectures, and quantitative relationships.* Each of the students was able to successfully construct and generalize an equation that described the functional relationship between the angle of the vertical displacement and the horizontal force. The evidence of the interviews suggests that students saw the tangent function as representing the relationship between the covariation of the angle and horizontal force rather than as a rule for the pointwise correspondence between the angle and the force. The students frequently focused on the boundary conditions of the problem to confirm their conjectures, as well as using geometric arguments to confirm the algebraic relationships created from curve fitting their data.

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WRITTEN SUBTRACTION IN MOZAMBICAN SCHOOLS

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Written subtraction procedures, used by Mozambican teachers and pupils are quite an interesting phenomenon. In the first place, from the point of view of the expressions that are spoken explicitly, as compared to the ideas that are used mentally. Secondly, from the point of view of the influence of particular educational and didactical traditions.

In this paper we present information on written subtraction algorithms used in Mozambican schools and some of our findings on how a group of 42 primary teachers do subtraction and interpret the procedures. They are students in a Master's Degree programme of Primary Mathematics (short: LEMEP) and have an average of 10 years of teaching experience.

1. INTRODUCTION

In Mozambican schools, the most commonly used algorithm of written subtraction is the following:

	in Portuguese:		*translated into English:*
	say (or think):	write:	say (or think):
762 - 238	oito para doze, falta quatro; vai um;	4	eight to twelve, missing four; goes one;
524	um mais três, quatro; para seis, falta dois;	2	one plus three (equals) four; to six, missing two;
	dois para sete, falta cinco.	5	two to seven, missing five.

It is similar to the procedures that are officially prescribed or recommended (or should we say: "traditionally taught"?) in Germany and France, apart from Portugal, from where the procedure was inherited since colonial times (Padberg 1986, p. 138-163; Eiller et al. 1980, p. 85-87; APMEP 1979, p. 35, 65-66; Queirós 1964, p. 39-42).

The Mathematics syllabi resulting from the first curriculum reforms in independent Mozambique (1975 and 1977, cf. Draisma et al. 1986) are so general that they give no details nor recommendations on the kind of written subtraction procedure(s) to be taught. In 1983 started the implementation of a major educational reform, with the gradual introduction of the National System of Education, with changes in the structure of the educational system and a corresponding curriculum reform involving all subjects of general primary (grades 1 - 7) and secondary (grades 8 - 12) education.

The authors of the revised Mathematics syllabus considered that the subtraction procedure traditionally used in schools: "*had two negative aspects:*

a) the majority of the teachers had great difficulties in explaining all details; in particular, in justifying the fact that you add (see the earlier example) 1 + 3 = 4, before subtracting 6 – 4 = 2.

b) the expressions spoken during the calculation are different from the standard expressions used for subtraction and addition; ...

Therefore the new syllabus introduces a slightly different procedure, based exclusively on:

— the notion of subtraction as inverse operation of addition
— the written algorithm for addition,
without introducing non-standard expressions:

EXAMPLE: computing 762 – 238 means finding the number that added to 238 gives 762:

	say (or think)	*write*
762	*eight plus four equals twelve; retain one*	*4*
– 238	*one plus three equals four; four plus two equals six*	*2*
524	*two plus five equals seven*	*5*

The explanation on the one you have to retain, which you later add to the three, is the same explanation given for addition." (cited from Draisma et al. 1986)

It is thought that the explanation of the written algorithm of addition is much easier for teachers and pupils, than any algorithm for subtraction (APMEP, w.d., p. 66). In this way, the teachers dispose of an easier explanation and the problem of the non-standard expressions is resolved. However, Padberg (1986, p. 139) points to the following disadvantages of using the language of addition, when doing subtraction:

- The sequence of the terms in the *spoken* subtraction shocks with the order in which the terms are *written*: in the spoken subtraction, the difference comes in between the subtrahend and the minuend (or sum), whereas in the written subtraction, the difference comes in the end (on the bottomline).
- Specific mistakes will result from pupils mixing up the algorithms for addition and subtraction.

At the time of the elaboration the new syllabus, no systematic research findings existed on the reality of learning and teaching Mathematics in Mozambican schools: Mozambique was too young a country and before Independence, curriculum, syllabi and schoolbooks were just imported from Portugal. The time schedule for the introduction of the new syllabus was so tight that it did not permit prior research nor experimentation.

2. WRITTEN SUBTRACTION BY THE LEMEP STUDENTS

Within the framework of my activities as lecturer for Didactics of Arithmetic, the LEMEP students answered two questionnaires on their habits of written subtraction, and on the way in which they learned the procedure.

The first questionnaire was presented to the students after three general discussions on different ways of doing subtraction with multi-digit numbers:

a) A general discussion on how the students do subtraction. During this discussion, the arithmetical practice of several students was analyzed.

b) A second discussion, based on a broad classification of subtraction algorithms into five categories (cf. Padberg 1986, p. 145), through the combinations of *two ideas of subtraction* (taking away or complementing), with *three techniques of "carrying"* (decomposition of the minuend, equal addition (Hughes 1986) or adding up):

c) A third discussion, on the way in which a Portuguese author on didactics of arithmetics presents the written algorithm of subtraction (Queirós 1964), distinguishing three ways of doing subtraction:

- — subtraction with borrowing (in our terminology: taking away combined with decomposition of the minuend);
- — subtraction with double borrowing (in our terminology: taking away combined with equal addition);
- — the customary (Portuguese, and therefore also Mozambican) way of doing subtraction.

The aim of the preliminary discussions was to prepare the students for a thorough analysis of their own habits of subtraction and the way in which these habits were formed.

Our hypothesis was that the Portuguese educational traditions constituted a strong unifying element in Mozambican teachers' ways of thinking.

The results of the questionnaire were surprising:

a) We found a greater variety of methods than we had imagined, corresponding to different algorithms that have been taught, in the past, in Mozambican schools.

b) On the notion of subtraction:

- some students always use the idea of taking away (see the example of Mussa)
- some students always use the idea of complementing or adding up (see the case of Tomé)
- the idea of subtraction used in the mind. not always corresponds to the expressions that are spoken: some students use the expression "para" —

related to adding up — but with the meaning of taking away (the case of Mussa); other students use the expression *menos* (= *minus* in English), but with the meaning of complementing or adding up (the case of Tomé)

- several students use both notions of subtraction in the same multidigit computation, changing from one notion to another according to the existence or absence of the necessity of *carrying* (the case of Pedro)
- several students use only one notion of subtraction (taking away or adding up), but when verbalizing the computation, they may change from one expression (*menos*) to another (*para*) (the case of Mussa)
- only one student changed his personal way of doing subtraction, because of the suggestions contained in present day syllabus, i.e., doing subtraction as an inverted addition, using the language of addition
- some students mentioned that they had to memorize, at primary school, only the basic facts of addition — because the subtraction algorithm was based on adding up, whereas other students had to memorize separately the basic facts of addition and the basic facts of subtraction.

c) On the techniques of *carrying* (or *transport*, as is said in Mozambican schools)

- notwithstanding the great variety of notions of subtraction that are used and accompanying expressions that are pronounced, nearly all students (40 out of 42) will compensate the necessity of *carrying* by adding one to the next position of the subtrahend.

  That means, that, in our example, after obtaining 4, in the position of the units, by different methods, in the position of the tens, nearly all will calculate: $1 + 3 = 4$ combined with $6 - 4 = 2$. Only two students have the habit to calculate: $6 - 1 = 5$; $5 - 3 = 2$.

$$\begin{array}{r} 762 \\ \underline{-\ 238} \\ 4 \end{array}$$

d) On the (mathematical) justification of the technique of *carrying*

- a considerable group of students admitted that they never had thought about the justification of the techniques they use; therefore, their justification is a result of recent reflection
- we noted a strong tendency of the students to justify now their technique of *carrying* by the "property of equal addition", which was mentioned by very few students in their answers to the first questionnaire. In this way, they avoid having to justify the change of the next digit of the minuend, which is considered a rather complex operation (APMEP p. 65, 66; Hughes 1967 p. 115), but, in order to do so, they need a formal property of subtraction that is not easy to teach either.

e) On the advantages of the traditional, Portuguese, expressions

- some students use the alternation of "menos" and "para" in order not to forget to "pay back" something in the next position
- some students use the alternation of "menos" and "para" in order to distinguish different methods of computation: one method within the limit of ten, another in the cases of the passover of ten
- several students showed how the use of "para" accelerates (abbreviates or "animates") the computation
- the use of "para" gives a possibility to read the terms of a subtraction in a non conventional order: *subtrahend*, *minuend* (= sum), *difference*, overcoming the clash noted by Padberg, between the spoken and the written subtraction.

3. SOME EXAMPLES FROM THE QUESTIONNAIRES

3.1 Mussa

$$\begin{array}{r} 583 \\ \underline{-\ 47} \\ ? \end{array}$$

"a) From three it is not possible to take away seven.

b) Therefore, I borrow; I take one ten from the eight and add it to the three in order to make thirteen.

c) Now I calculate: thirteen minus seven, six.

d) After that it's the turn of the "goes one". This one is the ten that I took from the eight. I say "goes one", in order not to forget that I took something from the eight.

e) Now I have to take away, to subtract, in reality, the ten from the eight; because I took it, but the eight is still the same, as if I had not taken anything.

f) Take one from the eight may be done in two ways: either take it directly, i.e.: eight minus one, seven, and continue the operation: seven minus four, three; or add the ten to the other tens of the subtrahend (one plus four, five) and continue the operation: eight minus five, three.
This is explained by the property: $8 - 1 - 4 = 8 - (1 + 4)$

During subtraction, I use more often the expression "menos"; sometimes I use "para", but not with a meaning different from the idea of taking away. The expression "para" appears to me only as a way of changing the language, maybe as a way of animating the computation.

In general, I use the expression "para", when, at a certain moment in the computation, I borrow from the next position and in order to take away this unit, I choose to add to the next position in the subtrahend. So I say, for instance: "one plus four, five; five to eight, three. It is here that the computation becomes animated: the expression "para" serves only to substitute the the expression "menos", without giving the idea that I changed my idea of computation.

The teachers, who taught me, always used the notion of "taking away". I don't remember having heard, from them, in primary school, the expression "para". I started to use the expression "para" at secondary school, without any explanation, only as an expression that may substitute the expression "menos"."

3.2 Tomé

"Usually, I do subtraction in the following way:

	in Portuguese:	*translated into English:*
583 $\underline{-\ 47}$	Treze menos sete, igual a seis; vai um.	Thirteen minus seven, equals six; goes one.
	Um mais quatro, cinco. Oito menos cinco, três.	One plus four, five. Eight minus five, three.
	Cinco, cinco.	Five, five.

I do use "menos" in order to safeguard the concept of the operation, because "menos" means subtraction. I say "menos", because of the sign. But in my conscience, in order to solve the operation, either I count forward from the bottom number until I reach the upper number, or, by trials, I do the sum of the bottom number with a digit number that may give the upper number. I verify whether I got the correct number, doing that addition, but in my conscience I know that I am doing a subtraction. When I do subtraction, I never use the expression "para".

In my external behaviour I use the term "menos", but my internal behaviour is complementing. I don't know whether you can speak of a combination of behaviours. In any case, I think that I use the notion of *complementing* and not that of *taking away*."

3.3 Pedro

"Using the problem 583 – 47, I do the written subtraction in the following way:

$$\begin{array}{r} 583 \\ \underline{-\ 47} \end{array}$$

a) I see that the process has to be position by position.

b) I see that from the digit in the position of the units (3), I can't take the digit 7 of the subtrahend; so I invent a ten which together with the 3 makes 13.

c) Then I operate: thirteen minus seven, six; goes one. This one is added immediately to the four — an automatic action — making five. This five originates the operation "five to eight, three".

d) In the end I take down the five. The result is 536.

In practice, I use the expression "menos" in the position where the operation originates a "vai um". Without the "vai um", I frequently use the expression "para", except for the case when the result is zero; then I use "menos". For me, "menos" and "para" have different meanings: "menos" emphasizes the idea of taking away and "para" emphasizes the idea of complementing.

The expression "para" is convenient in situations that don't lead to "vai um", because the numbers involved are smaller. E.g., in the case of "five to eight" you see the result three without a big mental effort. The parts that make up eight are recognizable. They are within the limit of ten."

4. CONCLUDING REMARKS

a) Although the influence of the Portuguese educational tradition is unmistakable, the interviewed teachers show many personal ways of using and interpreting this tradition.

b) Approximately half of the students use regularly the notion of adding up in order to find the difference of two numbers.

c) Separate studies done in Mozambican schools show that in grades 1 – 3 finger counting predominates and that doing subtraction by counting up is a common phenomenon for children who discover that counting-all strategies may be abbreviated (Kilborn 1991, Draisma 1992). That means that there is a considerable educational tradition that supports the use of an adding up algorithm for multidigit subtraction, just as is advocated by Fuson 1986 and 1988. Kilborn's proposal to substitute, in Mozambique, the adding up algorithm by what he calls "the modified decomposition method" (a *taking away* procedure), seems to be premature. His characterization of the adding up algorithm as "one of the most complicated there is" (Kilborn 1991, p. 77) lacks fundamentation. The attitude of the authors of the present day syllabus, of "not trying to change the existing traditions, unless you have strong arguments and the means to implement such a change with success" seems to be more appropriate for the moment.

e) An important complement of the present study will be the confrontation between subtraction verbalized in Portuguese and subtraction verbalized in Mozambican languages. Very probably, the computational habits of some of the students will have been influenced by their mothertongues, just as is the case of concepts of multiplication (Frouke B. Draisma 1995) and mental addition and subtraction influenced by Bantu numeration systems (Draisma 1993a, 1993b). The first data for this confrontation have been collected, but their analysis will take some time: the 42 students use 15 different mothertongues.

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CHILDREN'S REASONING IN SOLVING NOVEL PROBLEMS OF DEDUCTION

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This paper proposes an alternate theory for addressing children's reasoning in solving novel mathematical problems of deduction. It draws upon psychological theories of mental models, new ideas on the relational complexity of mathematical tasks, and recent studies of children's mathematical problem solving. The theory is applied to an analysis of children's reasoning in solving hands-on and written problems in which they must process a number of premises ("clues") to determine a solution. Two case studies are presented to highlight the different reasoning processes children apply to these problems, including how they interpret the premise information, whether and how they integrate premises, whether they consider alternate possible models during solution, whether they check and validate their models, and how they use external representations.

"Mathematics as reasoning" is listed as one of the major curriculum and evaluation standards for school mathematics, as recommended by the National Council of Teachers of Mathematics (USA, 1989). Informal deductive problems of the types shown in Fig.1 (and in the appendix) can be effective in developing children's mathematical reasoning and are frequently cited in curriculum documents (e.g., Australian Education Council, 1990; Baroody, 1993; NCTM, 1989). These informal deductive problems not only introduce children to the elementary ideas and processes of logic but also present them with novel problem situations in which they must develop their own reasoning strategies, in contrast to applying taught rules or algorithms. Furthermore, a recent study (English, in press a) has shown that children enjoy problems of this type. When given a selection of routine and non-routine problems from which to choose the problem they would most like/least like to solve, 25 out of 50 grade 5 children selected the problem in Fig. 1 as the one they would most like to solve.

Four famous sports people entered a television studio. One was a tennis player, one was a swimmer, one was a golfer, and the other was a chess player. Use the clues to find out who played what sport. Clues: Mr Bowler is not good at chess. * Both Mr Big's and Ms Ace's sports involve a ball. * Ms Fish can't swim at all. * Neither Ms Ace nor Ms Fish play tennis.*

Fig.1 An Informal Deductive Reasoning Problem

Scant attention however, has been given to children's approaches to solving problems of this nature. We know little about how children interpret these problems, the nature of the mental models they construct, how they reason with these representations, and how they deal with the complex relations entailed in problem solution. Many of the established psychological theories addressing deductive reasoning are of little assistance here as they focus on formal, hypothetical situations and frequently assume the application of formal rules of inference (e.g., Macnamara, 1986; Dias & Harris, 1990; Overton, 1990). Likewise, several of the existing theories of mathematical problem solving appear inadequate here, mainly because they fail to consider the complexity of the mathematical relations children must process during solution (e.g., Charles & Silver, 1989; Silver & Marshall, 1990). It is argued that the difficulty of representing relations has a major impact on children's mathematical thinking and that children's success with these deductive problems rests largely on their ability to deal with the complex relations entailed in solution (English & Halford, 1995).

This paper proposes an alternate theory to address children's reasoning in solving informal deductive problems. It draws upon psychological theories of mental models (English & Halford, 1995; Halford, 1993; Johnson-Laird, 1983; Johnson-Laird & Byrne, 1989,1991; Kintsch, 1986), new ideas on the relational complexity of mathematical tasks (English & Halford, 1995; Halford & Wilson, 1996), and recent studies of children's mathematical problem solving (e.g., Davis, Alston, Maher, & Martino, 1994; Davis & Maher, in press). As such, the theory has scope for addressing children's reasoning with a range of mathematical problems. In the remainder of this paper, I present an overview of the alternate theory, then describe a study examining children's reasoning with these deductive problems, and finally, apply the theory to a brief analysis of two case studies.

An Alternate Theory of Children's Informal Deductive Reasoning

The theory posited here involves three phases of problem solving, which modify and extend Johnson-Laird's stages in the process of deductive thought (see Johnson-Laird & Byrne, 1989, 1991). In the first phase, that of problem comprehension, reasoners apply their linguistic and general knowledge to construct an initial mental model of the information presented in each of the premises. It is argued that this comprehension phase entails the construction of two models, firstly, a problem-text model and secondly, a problem-situation model (English & Halford, 1995). The problem text-model is the mental representation the reasoner constructs from an initial interpretation of the verbal formulation (spoken or written) of the premise (cf. Kintsch, 1986; Kintsch & Greeno, 1985). The construction of this mental model involves extracting the relevant relational information from the text of the premise in an effort to make a meaningful interpretation of the relations present. This interpretation involves a transformation process in which the inherent relations are converted into a more explicit and workable form. For example, interpreting the second premise of the problem in Fig.1 ("Both Mr Big's and Ms Ace's sports involve a ball") entails recognizing that, of the sports listed, both tennis and golf involve a ball. The premise could then be transformed into the relational statements, Mr Big -> tennis or Mr Big -> golf, and Ms Ace -> tennis or Ms Ace -> golf. These represent simple unary or one-dimensional relations, which comprise just one source or dimension of variation. However other examples I address later involve the more complex binary (2-dimensional) and ternary (3-D) relations (Boulton-Lewis, Wilss, & Mutch, 1994; Halford, 1993; Halford & Wilson, 1996; Mayberry, Bain, & Halford, 1986).

The foregoing transformation process is a crucial step in problem solution because it is responsible for the construction of an appropriate problem-situation model which gives meaning to the premise and paves the way for the subsequent application of reasoning processes and problem-solving strategies. The construction of these problem-situation models is thus particularly important to the solution of these and many other mathematical problems (e.g., computational word problems, as discussed in English (in press b) and English & Halford (1995). It follows that children who form inappropriate models will have considerable difficulty in solving the problems, irrespective of the reasoning processes they employ.

The second phase of the theory is posited as the period in which significant deductive reasoning processes and problem-solving strategies are applied. These are responsible for planning a method of attack, for constructing integrated problem-situation models that will lead to a tentative solution model, and for overseeing the solution process itself. More specifically, for the deductive problems in question, this phase entails:

1. Choosing premises that most readily yield productive problem-situation models (such models would enable subgoals to be reached in solving the problem),
2. Of the premises that must be integrated, selecting a premise that can be integrated most readily with existing models,
3. Recognizing when more than one problem-situation model or tentative solution model is possible, and
4. During the course of problem solution, checking and verifying the construction or revision of problem-situation models.

A major component of the reasoning processes in this second phase is dealing with the complexity of the relations involved in integrating the premises. This can impose a considerable processing load if the premises themselves comprise complex relations, as indicated in subsequent discussion. While each of the premises in the problem of Fig. 1 comprise simple unary relations, the integration of appropriate premises entails ternary relations (i.e., three sources of variation). For example, we need to consider the two unary relations of premise 2 *in conjunction with* the two unary relations of premise 4 in order to work out who plays tennis (i.e., we need to consider all of the relations, Mr Big -> tennis or golf, Ms Ace -> tennis or golf, Ms Ace -/> tennis, Ms Fish -/> tennis). The complexity of premise integration is also increased when alternate models must be considered.

There is likely to be considerable back-and-forth movement between these first two phases, and within the second phase, during the course of problem solution (cf. Davis, 1984; Davis et al., 1994). It is argued that the child's transitional actions between, and within, these two phases play a fundamental role in problem solution. The third phase of the theory entails validation or verification of the final, solution model produced. The processes involved here are likely to be of a metacognitive nature where the conclusion is validated by reconstructing the final model. This may involve either reworking the problem to see whether the putative conclusion is again produced or "working back" from the final solution model by comparing it against each of the premises to ensure all conditions are met.

Nature of the Investigation

The participants were 264 children in grades 4 through 7, from state and non-state schools. Each child was individually administered two sets of problems (displayed clearly on cards), one set with hands-on materials (problem nos 1-3 shown in the appendix) and the other in written format only (problem nos 4-6). As the child worked each problem, he/she was asked to "think aloud" and explain and justify decisions made. Each child's verbalizations and procedures were transcribed and coded for subsequent analysis, which included: 1. the ways in which children interpreted the premise information, 2. whether children integrated premises and their effectiveness in doing so, 3. whether children considered alternate possible models during the course of solution, 4. whether children monitored their actions during solution and validated their final solution model, and 6. how effectively the children used external representations.

Selected Findings

Quantitative analyses showed main effects for grade with respect to several of the above variables on most of the problems. However these main effects did not reflect increased sophistication of reasoning processes across the grade levels. Rather, the main effects were due primarily to the unexpected responses of the older children, in particular, the sixth graders. In contrast to their younger counterparts, these children displayed little premise integration, rarely considered alternate models, and rarely used checking and validation processes. On the other hand, the majority of these children displayed superior use of external representations in solving the problems (e.g., made use of a matrix or table), that is, they used these representations to effectively reduce the complexity of the reasoning processes involved in problem solution (the external representations would enable relations in the premises to be considered serially which is less taxing than simultaneous processing; Halford & Wilson, 1996).

While there were no major differences among the remaining grades in their ability to integrate premises, there were significant differences in their consideration of alternate models during problem solution and in their use of checking, monitoring, and validation processes. On most of the problems, the younger grades (4 and 5) displayed superior reasoning here. Children in grades 6 and 7 rarely appeared to recognize alternate possible models, used little or no monitoring, and rarely validated their final solution model. Had they done so, they might have been more successful on the difficult problems (nos 3 and 6). Nevertheless, the grade 7 children were the most successful of all grades on problem 6 (74% of grade 7 children solved the problem), while the grade 4 children were least successful on problems 3 (55%) and 6 (45%). There were no other significant differences across the grades with respect to correctness. At least 60% of children in each grade were able to solve each of the remaining problems.

Greater insights into the children's reasoning processes can be gained by considering the protocols of individual children. Consideration is given here to Thomas, a grade 5 student, and Anthea, a grade 7 student. Both children were rated by their teachers as high achievers in mathematics, including competent problem solvers. Their responses to the most complex of the problems (problem 6, see appendix) serve to illustrate their contrasting reasoning processes.

Thomas gave this explanation as he worked the problem: *I'm putting down, well, each time I'm sure of where someone lives, like for the second clue it says the Jones live in the second house on the left. So that would mean they are in the second from the left. So I'm putting down five houses, just pictures of them, and in each house I'm going to write the person's name who lives there and so far I've got the Jones in the second house from the left.*
And it says the Wilsons live somewhere between the Taylors and the McDonalds. And it says the Jones live beside the Taylors, so that means the Taylors could either be on the left or the right of the Jones, but it says the Wilsons live somewhere between the Taylors and the McDonalds. So Taylors are probably first and the McDonalds probably last (fifth) and it says the Smiths live beside the Wilsons but not beside the McDonalds, so they'll have to .....well the McDonalds would be the last one (fifth) and the Wilsons would be the second last one (fourth) and the third last one, well the middle one, would be Smiths, and the first one would be the Taylors. The Smiths are in the middle house.

Thomas demonstrated quite sophisticated reasoning processes as he dealt with the complex relations inherent in this problem. He recognized the importance of commencing with a premise that would enable an explicit model to be constructed. Notice how he transformed his initial reading of this premise (i.e., his problem-text model) into a model that was more meaningful to him (i.e., his problem-situation model: "*It says the Jones live in the second house on the left. So that would mean they are in the second house from the left;*" such transformation is evident in his interpretation of other premises). He promptly recorded this information on his diagram. Thomas also noted that alternate models could be formed for the Jones and the Taylors. He retained this information mentally rather than recording it. In determining which model was correct, Thomas took premise 3 into account and made a tentative decision that the Taylors were to the left of the Jones. This would have entailed the processing of a complex quaternary relation, that is, a simultaneous consideration of the two binary relations, [RIGHT (Jones, Taylors), LEFT (Taylors, Jones)], and the ternary relation, [BETWEEN (Taylors, Wilsons, McDonalds)]. Thomas then confirmed his decision that the Taylors were to the left of the Jones by taking into account premise 1. This again would have involved the processing of a quaternary relation, namely, the binary relation, [LEFT (Taylors, Jones)], in conjunction with the ternary relation, [BESIDE (Smiths, Wilsons), NOT BESIDE (Smiths, McDonalds)]. The processing of these relations enabled him to complete his final solution model. Notice however, that Thomas did not validate this final model.

Anthea presents an interesting contrast to Thomas, despite the fact they were both classified as high achievers. She drew a simple illustration of five houses but this was of little benefit because she applied ineffective reasoning. Unlike Thomas, she commenced the problem with an inappropriate premise. After drawing her houses she immediately recorded the information of the first premise, that is, she wrote: SMITHS WILSONS ...., and explained: *I put the Smiths in the first house and the Wilsons in the second house because it says the Smiths live beside the Wilsons but not beside the McDonalds.* However she quickly noted her mistake and changed her drawing to: .... SMITHS WILSONS. She explained: *Because it says the Jones live in the second house on the left.* Anthea subsequently added the Jones in the appropriate position on her diagram. After considering the next (third) premise, she placed the McDonalds in the first position, explaining: *Because it says the Wilsons live somewhere between the Taylors and the McDonalds.* After considering the final premise, she completed her diagram: McDONALDS JONES TAYLORS SMITHS WILSONS, and explained: *It says the Jones live beside the Taylors. That means the Taylors are in the middle house.*

Anthea did not bother to check her final model. Her inability to solve the problem was due to a number of factors, including her failure to choose a suitable commencing premise, her avoidance of premise integration (i.e., she processed the premises in a serial fashion), her failure to recognize alternate possible models, and her construction of inappropriate problem-situation models (as evident in her interpretation of the Wilsons living between the Taylors and McDonalds). These inappropriate models might have been the result of her inability to entertain the more complex, ternary relations in the first and third premises; instead, she appeared to consider them as two separate binary relations (i.e., [BESIDE (Smiths, Wilsons) and then later, [NOT BESIDE (Smiths, McDonalds)]).

Concluding Points

This paper has proposed an alternate theory to account for children's reasoning in solving novel problems of deduction. The theory addresses the mental models children construct, the reasoning processes and problem-solving strategies they apply, and how they deal with the complexity of the relations involved in problem solution. The theory was applied to an analysis of grades 4 to 7 children's solutions to six novel deductive reasoning problems. While there was considerable variation in the nature of the children's reasoning processes, there was not an increase in sophistication of reasoning across the grades. There was, overall, little difference in the children's ability to integrate premises, with the exception of the grade 6 children who displayed a considerable lack of premise integration. The grade 6 and 7 children also showed little consideration of alternate models during solution and rarely applied any checking, monitoring, or validation processes. It would seem that the older children did not see the need to apply these important processes when they had efficient external strategies at their disposal. Their responses highlight the importance of actively encouraging children's continued development of mathematical reasoning across the primary grades. In particular, we need to ensure that children do not rely solely on specific skills and procedures at the expense of important reasoning processes.

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Appendix

SET 1:HANDS-ON PROBLEMS

Problem No.1 *Materials*: Four cards with the names, CARLA, BILL, JACK, SALLY, placed randomly before the child and 5 upright wooden bears with the outfits: green top/blue pants; blue top/orange pants; green top/orange pants; yellow top/blue pants; yellow top/orange pants. *Problem:* Carla, Bill, Jack, and Sally went to a party and they each wore a different outfit. Use the clues to find these bears amongst the group of bears.*Clues:* * Orange pants are the favourite of Carla and Sally. *Jack prefers blue pants to orange pants. *Bill likes green tops but Jack doesn't like green tops.* Carla does not like green tops or yellow tops and Sally doesn't like blue tops or green tops. * Bill likes either blue pants or orange pants.

Problem No. 2 The child was to build a tower using the following clues and the blocks provided. *Clues:** The red block goes just below the yellow block.*The blue block goes just above the white block and just below the red block. * The yellow block is somewhere between the green block and the white block.

Problem No.3 The child was presented with five playing cards (king, jack, ten, queen, ace) and instructed to arrange them according to the clues: *The jack is immediately to the right of the queen.*The king is to the right of the ace. *The queen is somewhere between the ten and the ace. *The ten is immediately to the left of the queen.

SET 2: WRITTEN PROBLEMS

Problem No.4: See Fig.1

Problem No.5 Bill stacked five of his school books on his desk. Use the clues to find which book is second from the bottom. *Clues*:*The reading book is touching the dictionary. * The reading book is just below the library book. *The atlas is touching the library book but not the science book.

Problem No.6 There are five houses along one side of a street. Use the clues to work out who lives in the middle house. *Clues*: *The Smiths live beside the Wilsons but not beside the McDonalds.*The Jones live in the second house on the left. *The Wilsons live somewhere between the Taylors and the McDonalds. *The Jones live beside the Taylors.

ASSOCIATION JUDGEMENTS IN THE COMPARISON OF TWO SAMPLES

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SUMMARY: *In this research an experimental study is introduced about the strategies and the association judgements used by students when they face to a problem about comparison of a numerical variable in two different samples (comparison of two samples). The strategies are classified from a mathematical point of view so we are allowed to identify theorems in action and two kinds of misconceptions about association.*

The importance of the association concept within the Statistics cannot be discussed since it's the base of a lot of statistical methods. Moreover it must be present in all pre-university and university courses about the introduction to Statistics. The judgements about relations or covariations (association judgements) among events are a critical aspect of the human knowledgement Crocker (1981).

The approach about the initial conceptions - preconceptions - (Artigue,1990, Confrey, 1990) and about the strategies used by the students when they face to an association problem is very important from the didactic point of view since it would help us to plan the teaching. It's also very important to identify concepts and theorems in action in the same way as Vergnaud (1982) who thinks *"The essential purpose for a cognitive analysis of tasks and behaviors is to identify such theorems in action"* (pp. 35).

This study is included in a wider approach about the statistical association (Estepa 1994; Estepa and al., 1994; Estepa and Batanero 1995; Batanero and al. (in press), where they study the initial conceptions, association judgements and resolution strategies used by the students in contingency tables and in scatterplots. Through this paper we'll study the comparison of two samples.

In previous articles (Estepa and Batanero, 1995, Batanero and al. (in press)) we have identified misconceptions about the statistical association such as: Determinist conception of association: Some students do not admit exceptions to the existence of a relationship between the variables. They expect a correspondence which assigns only a value in the dependent variable for each value of the independent variable. Localist conception of association: Students often form their judgement using only part of data provided in the problem of association. If this partial information serves to confirm a given type of association, they adopt this type of association in their answer. Often this partial information is reduced to only one conditional distribution or even only one cell, frequently the cell for which the frequency is maximum (in contingency tables). This type of conceptions are also shown when pupils face to problems about samples comparison.

Sample

The sample consisted of 213 students in the last year of secondary school (18 years old students). It is in this level where the topic association is introduced in the Spanish syllabus. The questionaire was given to the students before the instruction was started. About half of the students (113) were males and half (110) females. This study has a quasi-experimental character, because of the non-random character of the samples of students and problems

Questionnaire

The students have been asked two questions about the samples comparison shown in figures 1 and 2.

Item 1. When the blood pressure was measured before and after applying a certain medical treatment to a group of 10 women they got the following values

	Blood pressure in every woman									
Woman	Mrs A	Mrs B	Mrs C	Mrs D	Mrs E	Mrs F	Mrs G	Mrs H	Mrs I	Mrs J
Before the treatment	115	112	107	119	115	138	126	105	104	115
After the treatment	128	115	106	128	122	145	132	109	102	117

Using the information contained in this table, do you think that for this sample the blood pressure depends upon the fact of measuring it before or after the treatment? Explain your answer

Figure 1

Item 2 The following data were obtained when the sugar level in the blood was measured in male and female schoolchildren

	Sugar level in the blood																			
Pupil	A	B	C	D	E	F	G	I	J	K	L	M	N	O	P	Q	R	S	T	U
Gender	M	M	M	M	M	M	M	M	M	M	F	F	F	F	F	F	F	F	F	F
Sugar level in the blood	9	0	9	8	6	7	4	9	8	9	6	0	7	0	8	3	6	7	7	3

M = male, F = female

Using this information contained in this table, do you think that for this sample the sugar level in the blood depends on the sex? Explain your answer

Figure 2

As we can see the samples of the first item are related among themselves and those ones from the latter are independent. For each one of the items a contrast of differences has been done by means of the test t getting p= 0.00092, t= 3.3 for the first and p= 0.1510, t= 1.5 for the second one. A sample study was done with a first sample of 53 pupils, which was useful to check the reliability, the codification system of the students answers and to improve the final version.

DISCUSSION

The data were collected and the students' answers were classified into categories. Two independent variables were considered: the existence or not of association between the two samples (association judgement) and the procedure used to take this decision. In table 1 it's shown the frequency and the percentage of answers about the existence of association in the comparison of two samples. For the pupils it's not a difficult task to detect the association among the samples.

Nevertheless, the correct judgement about the existence of association between the two samples is not enough to determine the preconceptions that the students have about this topic, since, as it has been demonstrated in different researches, we can get a correct solution of a certain problem using a incorrect procedure. From the Mathematics Education point of view it's necessary to use a correct procedure and the emission of a correct association judgement for an appropriate understanding of the idea of association.

Table 1 Frequency and percentage of answers about the existence of association in comparison of samples

Item	Associación	Independence	No answer
1	*178 (83.6)	20 (9.4)	15 (7.0)
2	*157 (73.7)	44 (20.7)	12 (5.6)

* correct answer

The emission of an association judgement of a quantitative variable with regards to a dichotomic (comparison of two samples) is a task shown in a number frame (Douady,1986). Furthermore, being the dependent variable quantitative we can compare several statistical of these distributions such as means and ranges. To evaluate the students' preconceptions it's necessary to take into account the association judgements as well as the strategies used. So we have classified the strategies according to the mathematical content put into practice -explicit or implicitly- into correct, partially correct and incorrect strategies.

CORRECT STRATEGIES

1. To compare means: The pupils compare the mean of the two samples to decide the existence of association among variables. This is an example of a correct answer for the institutional relation introduced to the student, which does not include the usual

procedures of contrast of equality of means, where as well as the value of the means, it's necessary to take into account the variability of the two sets of data, the size of the samples and the level of meaning (Ríos,1967). Implicitly the student is using the following theorem in action: "If the means of the conditional distributions of a quantitative variable with regards to another qualitative one are different enough it implies that there is an association among the variables" (T1).

2. To compare the total additions: Some students have compared the addition of the values of the variable in each one of the samples. So, they are using in a virtual way the property which says that the dependence will imply the variation of the statistics of the variable's distribution, which is a modification of the previous theorem in action T1.

3. To compare percentages: Other pupils have compared pairs of corresponding values indicating proportions or percentages of cases where the pressure increases or decreases. In this case, the distribution of frequencies is not reduced but it is really being produced an agrupation of the differences' frequencies among the variable's values in both categories. If there wasn't dependence, these both categories should have a similar frequency. This is the foundation of no parametric methods which are based on the comparison of signals or series (Siegel,1986). There are even some students who think that the solution of the problem is related to the probability and so they use Laplace's rule. The pupils are using the following theorem in action: "If once we've got a common value in both samples, the frequency percentages of the higher (lower) values are similar it implies the independence" (T2).

4. To compare the distributions: The students compare the distributions of frequencies of both samples, for example: *"I say that when we compare the males' data to the females' these latter have got lower blood pressure. All that is proved with the fact of two girls having 0, while there is only a boy with 0. There are 2 girls with 3, while there isn't any boy with that pressure, only 1 out of 4. There are 2 girls with 6 and one boy with 6. There are 3 girls out of 7 and only 1 boy out of 7. There are 2 boys out of 8 and only 1 girl out of 8. There are 4 boys out of 9 and no girl out of 9"*. They use the theorem in action: "There is independence if both frequency distributions are similar" (T3).

PARTIALLY CORRECT STRATEGIES.

5. To compare each one of the cases. The pupils compare the corresponding values in both samples. As in the strategy 3, but in this case proportions or percentages are not used, and so this strategy is only considered partially correct. They use the theorem in action: "The variables' dependence implies that the difference of the corresponding values in related samples must have no null mean" (T4). Some students have been wrong with regards to their association judgement because they hoped the variation did have the same sign in every case, which indicates a functional or determinist conception of the association.

6. To indicate exceptional cases. They use the cases where the general rule is not followed (exceptional cases) in order to determine the existence or not of association between the two samples. They don't use proportions or percentages. A localist conception can be seen because a little of the distributions is used.

7. To find out the differences.The students calculate the differences among the pairs of values, giving the association judgement according to the differences they've got. These pupils use the theorem in action: "In order to find association between a continuous variable and a dichotomic one the difference among the corresponding values of the continuous variable's conditioned distributions must be large enough" (T4).

8. Global comparison. It' s when the association judgement is carried out by means of a global comparison of both samples. For example, *"In the chart we can see that the males have got a higher sugar level with regards to the females"*. They use the following theorem in action: " If the global difference between both samples is large enough then there is association" (T5).

INCORRECT STRATEGIES.

9. To expect similar values. Sometimes in each one of the samples similar values are expected. As it doesn't happen because of the irregularity they can see on the data they conclude that the blood pressure is independent of the treatment, showing a determinist conception of the association.

10. To compare maximals and minimals of both distributions, basing the association judgement on these points, which indicates us a localist conception of the association.

11. To compare ranges. There are some students who compare the range of both samples.

12. To value coincidences. Some students justify the no dependence for the coincidence of several cases, just as the student who answers: *"Because they can have the same sugar level and they can be male or female, like for instance the pupils F and N who have got the same sugar level 7 and however they are different, male and female"*.

13. To expound previous theories. Some students base their association judgement on their knowledgement about the context (previous theories) instead of using the data given in the problem like this student's answer: *"It depends on the fact you are examined by a male or female doctor when you don't feel very well and at that moment you can get a bit excited and that's the reason of the increase of the blood pressure"*. In Psychology this phenomenon is called "Illusory Correlation" and it was studied among others authors by Chapman and Chapman (1969) .

14. Others. In this category other kind of procedures and strategies have been included, like how to compare the means, but calculating them without taking into

account 0, this phenomenon had already been indicated by Mevarech (1983), they use the variables' values as percentages.

The frequency and the percentages of the use of these strategies in the students' answers are shown in the table 2.

Table 2. Frequency and percentage of the strategies used by the pupils in the comparison of samples.

STRATEGIES	ITEM 1	ITEM 2
Correct		
1. To compare the means	2 (0.9)	26 (12.2)
2. To compare the total additions	1 (0.5)	35 (16.4)
3. To compare percentages	41 (19.3)	1 (0.5)
4. To compare the distributions		34 (16.0)
Partially correct strategies		
5. To compare each one of the cases	73 (34.3)	3 (1.4)
6. To indicate exceptional cases	63 (29.6)	
7. To find out the differences	6 (2.8)	
8. Global comparison		64 (30.0)
Incorrect strategies		
9. To expect similar values	3 (1.4)	1 (0.5)
10. To compare maximals and minimals		6 (2.8)
11. To compare ranges		3 (1.4)
12. To value coincidences	1 (0.5)	3 (1.4)
13. To expound previous theories	1 (0.5)	8 (3.4)
14. Others	8 (3.8)	14 (6.6)
15. No answer	14 (6.6)	15 (7.0)

CONCLUSIONS

The largest percentage of correct strategies appears in independence samples, while we can find a higher percentage of partially correct strategies in related samples.

An important thing to mention is that there are strategies characteristic of the related samples, like the strategies 3,5,6,7, and 9, while there are others which appear more strongly in independent samples, like the 1,2,4,8,10,11,12 and 13 ones. From all this we can conclude that even though the pupils haven't worked formally with the concepts of independent and related sample, they distinguish these concepts when acting on them trying to detect the existence of association among their values.

The pupils who use the comparison of means strategy in independent samples, consider the mean as a representative value of the data (Pollatsek and cols., 1981; Mevarech, 1983). During the research which was carried out by Leon and Zawojwski (1991), they select 4 properties of the mean taken from Strauss and Bichler (1988): A) The mean is found between the utmost values; B) The addition of the deviations with regards to the mean is 0. F) When the mean is calculated, if the value O is present, it must be taken into account; G) The mean's value is representative for the set of data from which that parameter is deduced. They think the properties F and G are more difficult to understand than A and B. The students, who have used the mean, have used Leon and Zawojewski's property G and so they would show a relational knowledgement of the mean (Skemp,1978, Pollatsek and cols.1981) which consists in having suitable schemes or enough conceptual structures to solve a wide range of problems where the concept is necessary. The pupils who compare the totals (16 4 %) would be also at a similar level to the ones who compare the means, since the data addition is a typical value of the set of data (Pollatsek and cols,1981).

The strategy which is based on a recount of the suitable and unsuitable cases (strategy 3) has transformed the original problem into a contingency table 2x 2 using a probability comparison.

From the results we've got we can infer that the comparison heuristic of the corresponding values can be considered as a strategy with different levels of elaboration since it can be seen in the 87.9 per cent of the cases.

When two samples are compared, one of the most frequent questions the researcher faces to is if those samples come or not from the same population. This is what, intuitionally, the students who compare distributions have done (strategy 4).

When the data are globally compared (strategy 8) the pupil is implicitly accepting the idea of all the data are relevant. To decide which data are relevant it's a basic step to do an association judgement to tell that judgement (Crocker,1981).

The determinist conception of the association appears when the student expects the data's variation is always in the same sense (strategy 5) or when he/she expects similar values in both samples (strategy 9).

The localist conception can be seen when the pupil uses cases to decide his/her association judgement either using the exception cases or comparing the maximals and minimals in both samples.

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ON SOME FACTORS AFFECTING ADVANCED ALGEBRAIC PROBLEM SOLVING

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This paper focuses on the learning of algebra at undergraduate level. Some general hypotheses on cognitive, didactical and linguistic factors affecting algebraic problem solving are discussed and tested. In particular, the duality process/object is taken into account in order to explain students' problem solving performances. Some empirical findings are presented concerning the resolution of a sequence of algebra problems by a group of freshman computer-science students over a four-months term. Students' results in the whole sequence are compared to their results in other subject matters.

1. Introduction

The teaching of mathematics to university students other than mathematics undergraduates raises new problems for researchers in mathematics education. This holds in particular for the teaching of algebra to computer-science students. The aim of this paper is to analyse students' difficulties in algebra learning and to check some interpretative hypotheses.

By 'algebra learning' I mean not only the learning of definitions, theorems and algorithms, but also the development of algebraic problem solving skills. The problems may involve structures only defined by abstract properties (e.g. problems like "... let G be a group such that ...") or already known structures, such as the standard number systems, real vectors, matrices and polynomials, and may require more or less mechanical application of rules within some notation system and more or less understanding of the meanings involved. The strategies required may be straightforward or rather complex. In particular, problems may involve representation processes, including translations between different semiotic systems. Moreover, problems may significantly require the application of already known mathematical facts and procedures.

To interpret students performances some general factors will be considered that are, in my opinion, adequate to account for most of students' difficulties in the resolution of problems such as those proposed as evaluation tests during an algebra undergraduate course and presented in 2.1..

First of all, as widely recognised in literature, mathematical notions can be conceived as *objects* or as *processes* (Sfard, 1991; Tall *et al.* 1991). A similar analysis has been carried out by Arzarello (1991), who points out the *polarity* between the *procedural* and the *relational* interpretation of the algebraic code. Such duality plays a major role in learning processes. It seems that many students cannot go beyond the procedural interpretation. Therefore problems even simple but requiring to handle relationships may result difficult, whereas the availability or feasibility of algorithms requiring little semantic control should produce some improvement of students' performances.

Such cognitive factors are strengthened by 'didactical' ones: students are accustomed (by their high school mathematical practice) to be required only to do something, mostly to perform few well-known algorithms. Moreover, the role of some mathematical subjects that are strongly stressed in the school curriculum, together with the related algorithms, is often overestimated by students. This implies that they are accustomed to activate only a small number of *conceptual frames* (Arzarello *et al.*, 1993) and try to apply them to any problem they have to solve. Therefore, problems requiring conceptual frames different from the standard ones should result more difficult.

A third factor is connected to language. As remarked by Laborde (1982), mathematical language (LM), which is used in school practice, is a combination of natural language and symbolic code. Actually, LM is used both to describe mathematical objects, procedures and relationships and to communicate within the classroom, according to everyday-life conventions. This implies that different conventions may conflict. For example, in ordinary communication contexts, 'two numbers' in most cases means 'two different numbers', whereas in mathematics it may often mean 'a number' as well. On account of this, the command of written (Italian) language and its structure should result an important factor promoting algebraic learning and problem solving. What I mean is that students able to understand and produce well-organized and semantically adequate complex periods (e.g. containing conditional time clauses) should better express mathematical relationships and handle the conflict between the different conventions involved.

2. A sequence of problems

2.1. Criteria of choice

In this paper I take into account 1 problem from the pre-test given on October 1994, 1st, and 4 problems taken from the tests given from October to January 1995. The pre-test was anonymous, whereas the other tests have been given for academic evaluation purposes. The pre-test and the other papers have been matched in May, 1995. Moreover, the pre-test papers have been used to evaluate students' linguistic competence, in a way described in 3.3..

The problems are different as to complexity of the problem situation and relationship between knowledge involved and procedures to be performed.

P0 *(October, 1st; first lesson of the course)*

50 students, out of a group of 100, passed Calculus, 60 passed Algebra and 70 passed Geometry. What can you state about the number of students passing all the three examinations?

P1 *(End of October)*

Take into account the function f: $N \times N \to N$, defined by $f(m,n)=2^m 4^n$. Would you agree that f is injective? If you do, prove it, if not, refute it.

P2 *(End of October)*
The statement "For any natural number n, the natural number n^2-n+11 is a prime number" is:

true ☐ false ☐ other ☐ ?

Explain your answer.

P3 *(End of November)*
Prove or refute the following statement: "Let a, b, c be integers. If a divides bc then a divides b or a divides c" (more formally, "$a|bc \Rightarrow (a|b \vee a|c)$").

P4 *(January)*
Find out a subgroup H of $(\mathbb{Z}, +)$ satisfying all of the following conditions:

(a) $8 \in H$; (b) $18 \in H$; (c) $1 \notin H$.

How many subgroups fulfil all of the conditions (a), (b), (c)?

The students involved in the pre-test are 45, those involved in the other tests are 32. 13 students left off attending the course during October.

2.2. A priori analysis

P0 has been given before the beginning of the course, after a two-week optional preparatory course, mainly focused on pre-calculus and calculus. From a mathematical viewpoint, it is very simple and might be solved even by young children, with the aid of manipulatives. Nevertheless, it cannot be solved within arithmetic nor within set-theory only, but requires a bit of both. There is no algorithm to be automatically applied and students have to build a suitable one. Moreover, the answer is not a number, but an interval. The lack of an algorithm working within a single notation system and providing a number as result could result a major source of trouble for a number of students.

From a procedural perspective, **P1** seems not very hard, as its complexity could be reduced by evaluating (m,n) and searching for a different couple (x, y) such that $2^m4^n=2^x4^y$, or even $m+2n=x+2y$. This procedure leads to an answer almost easily, if one does not take (m,n)=(0,0) or (m,n)=(1,0). Trial-and-error strategies might be used and some standard properties of the exponential function could make the search more effective. Anyway, even mechanical search procedures have good chances to get an answer. In other words, this problem may be solved by means of procedures requiring little mathematical control.

P2 (Arsac *et al.*, 1992) has been given just to convince students that the strategies based on a (little) number of numerical checks are very dangerous. Actually, the least n for which n^2-n+11 is not a prime number results 11.

P3 is different. A purely mechanical search for counterexamples might fail to get to an answer. For example, if one takes a=3 (or any other prime number), the statement is satisfied by any choice of b, c. The divisibility relation is expected to generate some trouble, as it is not a function and induces a partial, non-linear ordering on the set of natural numbers.

P4 can be divided into two parts. The simple construction of the subgroup seems not very hard. From a strictly procedural point of view it requires something more than P1 but much less than P3. Actually one has only to apply an already learnt rule, i.e. to compute the greatest common divisor of $\{8, 18\}$ and consider the subgroup generated by it. So the condition $1 \notin H$ is automatically fulfilled. The proof of the uniqueness of the construction is a different thing, because it requires algebraic knowledge. One has to stop any procedure, change his/her perspective and take into account a general representation theorem (i.e. "All the subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$, with $n \in \mathbb{N}$"), which has not been used in the construction. Students are requested to go beyond a subjective interpretation of uniqueness ("I cannot do anything else") and to regard uniqueness in an objective way ("I can prove that nobody could do anything else"); moreover, they should use knowledge (not only hands, pencil or pocket calculator) as a tool.

3. Outcomes

3.1. Data concerning single problems

P0 requires to find out the range of the number of students passing three examinations, x for the sake of brevity. The following tables show the distribution of answers, the strategies and the visual representation used, if any.

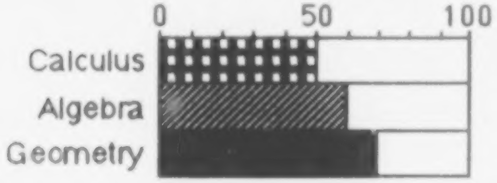
Table 1: answers to P0

ANSWER	N°	ANSWER	N°
$0 \le x \le 50$ (with a suitable explanation)	8	$30 \le x \le 50$	1
$0 \le x \le 50$ (with no suitable explanation)	9	$? \le x \le 50$	7
$10 \le x \le 50$	3	$x=50$	6
$20 \le x \le 50$	1	no answer	10

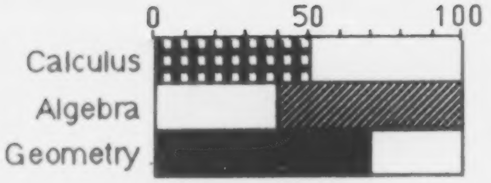
Table 2: strategies and representations for P0

STRATEGY	N°	CORRECT
A: no recognizable strategy or no attempt to find an answer	4	0
B: only based on verbal representations of the problem situation	21	5
C: introducing visual representations of the problem situation	13	3
D: probability (of passing all the three examinations: 21%)	2	(2)
E: seemingly random arithmetic operations with no explanation	3	0
F: seemingly random set theoretical operations with no explanation	2	0

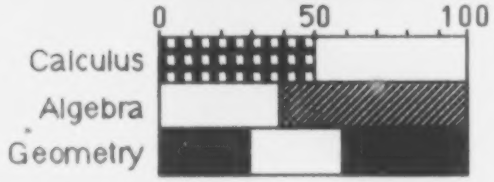
Table 3: visual representations for P0



Three left-aligned bars: 6 students; none of them gave a correct answer but all claimed x=50.



Two left-aligned bars and one displaced: 5 students; 1 of them provided a correct answer; the others claimed $10 \le x \le 50$ or $20 \le x \le 50$, according to the bar displaced.



One bar broken:: 2 students; all of them provided a correct answer.

In this particular experiment all the representations have been bar diagrams or something alike.

The correct answers to **P1** are 21. The remaining 11 students provide a wrong answer, or no answer at all. A problem analogous to P1 but involving the function defined by 2^m3^n has been solved in class,by students themselves, few days before. 4 students do not realise the difference between the two problems and argue from analogy that f is injective, as if 4 were a prime number. 4 students claim that f is injective because $m+2n=m'+2n' \Rightarrow m=m'$ and $n=n'$. Another claims that f is not injective because $m+2n=m'+2n' \Rightarrow (m=m'$ and $n=n')$ or $(m=2n'$ and $m'=2n)$. Answers like this have been classified wrong.

P2 has been correctly solved by 15 students. The wrong answers can be classified into two groups. 9 students generalise the outcomes of a small number of numerical checks (in most cases chosen among n= 1, 2, 3, 5) and conclude that the statement is true. 8 students study the expression n^2-n+11 as a polynomial , find that $\Delta=1-44<0$ and conclude that "it has no real roots and so does not split into factors".

The correct answers to **P3** are 16. Some wrong answers are consequences of undue generalisation. 4 students perform a little number of trials, generally with prime numbers, such as 2, 3, 5, and then conclude that the statement is true. 5 students, providing either correct or wrong answers, identify the relation "m divides n" with "$m \le n$". This interpretation leads 2 of them to find out the triple 6, 3, 2 as a source of counterexamples which are actually answer the question. 3 students misinterpret the implication and prove or disprove formulas different from the given one. 3 mistake "m divides n" for "m is divisible by n" or even "m divided by n". 6 students, providing both correct (2) and wrong (4) answers, introduce equations like km=n to express 'm divides n'.

The correct answers to **P4** are 20 and can be divided into two groups too. 9 students perform the correct construction but give no proof of its uniqueness. At this regard, they provide some reasonable arguments which are by no means conclusive. What they actually prove is the fact that the procedure they use could not lead to any different answer. 11 students use the representation theorem for subgroups of ℤ and are able to provide a complete, satisfactory proof. 7 students give no answer. The 5 students giving wrong answers show a poor command of algebraic language and do not succeed in handling the three conditions at the same time.

3.2. Language

The pre-test papers (related to P0 and other 4 problems) have been classified according to the mastery of language displayed. Three levels have been recognised.

Level 0 (L0): rambling words, ill-organised sentences. (10 students)

Level 1 (L1): well-organised, semantically adequate simple sentences; few compound, no conditional sentences. (24 students)

Level 2 (L2): a good number of well organised, semantically adequate compound sentences, including conditional ones. (11 students)

The students who left university during October have been 13; among them, 8 are L0 and 5 are L1. The students passing algebra before May, 1995 have been 25; among them, 11 are L2, 14 are L1. No L0 student has passed algebra before May.

3.3. Longitudinal analysis

The problems more explicitly connected with the algebra curriculum are P1, P3, P4. P2, which is actually connected to the curriculum, has been interpreted by most students in terms of their high school practice (and contract). The grouping of students according to their answers to the sequence P1-P3-P4 may be useful to point out their attitudes and styles. In each of columns 1-4, 1 means correct answer to the corresponding problem, 0 means wrong or missing; the column P4U refers to the proof of the uniqueness of the solution to P4.

Table 4: answers to the sequence P1-P3-P4

P1	P3	P4	P4U	N° of students			
				Total	L2	L1	L0
1	1	1	1	6	3	3	0
1	1	1	0	7	3	4	0
1	1	0	0	0	0	0	0
1	0	1	1	1	0	1	0
1	0	1	0	2	1	1	0
1	0	0	0	5	2	2	1
0	1	1	1	1	1	0	0
0	1	1	0	2	1	1	0
0	1	0	0	0	0	0	0
0	0	1	1	1	0	1	0
0	0	1	0	0	0	0	0
0	0	0	0	7	0	6	1

The 6 students with four '1' are the elite of the class: they not only can use procedures to solve problems, but also can use their knowledge to check their solutions. All of them got top marks in all the 5 first-semester courses. The 2 students with '0111' or '1011' belong to this group too (group A). The 7 students with '1110' (group B) can perform procedures correctly to solve problems but do not show a complete command of algebraic knowledge. The analysis of their papers shows that their problem-solving style is mainly rule-based; they can correctly check initial conditions and results of algorithms, but get in trouble when asked to use relational algebraic knowledge. The 2 students with '0110' belong to this group too. All students of group B have passed algebra before May but, with two exceptions, they have passed less examinations and with lower marks than group A. The 5 students with '1000' (together with the 2 students with '1010') can perform some easy procedure but display a poor control of them. They have passed at most 1 examination (out of 5) before May.

4. Discussion

The duality between operational and structural conceptions has proved an important tool to interpret students' problem solving behaviors. **P1** and **P4** (as only regards the construction of the subgroup), which can be solved by algorithms requiring little mathematical control result easier than other problems. **P0**, which to some extent could be considered easier, has proved more difficult because it asks for some semantical control. The strategies of students trying to apply a random sequence of operations within some notation system (arithmetic or set theory) are remarkable. Similar remarks hold for **P3**; the efforts of some students to replace 'm divides n' with '$m \leq n$' (introducing a linear ordering), or with '$km=n$' (and interpreting the symbol '=' operationally), point out the dominance of the 'arithmetical' problem solving style.

As regards **P2**, the expression n^2-n+11 has been interpreted in three different ways:

- operationally, as a function defined on the whole set of natural numbers;
- operationally, as a computational rule, with no attention to the domain;
- structurally, as a formal object (polynomial).

The third interpretation seems clearly affected by didactical factors. The same holds for some patently inadequate answers to **P0** (for example, students writing down random sequences of arithmetical or set-theoretical operations, or claiming that exactly 50 students have passed three examinations). This shows the attractive power of strongly institutionalized high school techniques as paradigms of doing mathematics. Most likely, the action of handling a polynomial has been regarded by students as contractually more valuable than the action of just computing a function. Nevertheless, this seemingly structural interpretation has been adopted only by average or high level students. In other words, structural interpretation results a crucial step even when wrong.

As regards language it is worthwhile to remark:

- all students passing algebra before May are L1 or L2;
- all L2 students passed algebra before May;
- bright students are almost equally distributed between L1 and L2.

Belonging to L2 seems a sort of insurance against failure. Nevertheless, it seems not a necessary condition for proficiency in algebra.

Further research is needed to develop some of the ideas sketched in this paper. In particular it seems necessary:

- ◊ to define and analyze more accurately the components of linguistic competence to be related to algebraic problem solving performance;
- ◊ to refine the analysis of algebra tasks from the viewpoint of the duality process/object;
- ◊ to extend the research to a larger sample;
- ◊ to single out the factors that induce students to activate specific conceptual frames.

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INTUITIONS AND SCHEMATA IN PROBABILISTIC THINKING

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Abstract

It was found that the evolution with age (grade) of probabilistic misconceptions is rather divergent. Some probabilistic intuitions improve with age, but others become worse in older subjects. This finding is explained by the tacit influence of some intellectual schemata on the structure of intuitions. Intellectual schemata, though strengthening with age may, sometimes, only apparently, fit the requirements of the given problem and consequently they may mislead the older subjects more than the younger ones.

Introduction

The main aim of the present research was to investigate the evolution with age of a number of intuitively based, probabilistic misconceptions. Our initial hypotheses, based on previous findings related to the intuition of infinity, was that intuitions, in general, become stable after the emergence of the formal, operational stage (see Fischbein, Tirosh, & Hess, 1979). It has been assumed that age and usual instruction may enrich the realm of knowledge, but cannot change the basic, intuitive attitudes of the individual. This assumption is contradicted by the findings concerning probabilistic misconceptions exposed in the present paper.

As far as we know, the evolution with age of probabilistic misconceptions has not been investigated, systematically, up to now. Our main finding has been that intuitive misconceptions develop rather divergently with age. The impact of some misconceptions diminishes with age, some become stronger, leading to higher frequencies of erroneous answers in older students, and some remain stable across ages.

Methodology

A number of well-known probabilistic misconceptions were investigated. In the present report, we describe only a part of a larger project.

Subjects

Four groups of subjects were investigated, corresponding to grade 5 (ages 10-11), grade 7 (ages 12-13), grade (ages 14-15), and grade 11 (ages 16-17). At each of the four school levels, 20 subjects participated in the research. None of the subjects received any previous training in probabilities.

Instrument

A questionnaire was drawn up consisting of seven probability problems and was administered according to the usual classroom conditions. Each session took about one hour. Because of space limitations we will describe in the present report, the findings referring to only four questions. Each question will be reproduced first in its original wording followed by the respective findings.

Representativeness (See Kahneman & Tversky, 1972) Question 1

In a lotto game, one has to choose 6 numbers from a total of 40.

Vered has chosen: 1, 2, 3, 4, 5, 6.

Ruth has chosen: 39, 1, 17, 33, 8, 27.

Who has a greater chance to win?

A. *Vered has a better chance to win.*

B. *Ruth has a better chance to win.*

C. *Vered and Ruth have the same chance to win.*

Explain your answer.

The expected, correct, answer is obviously C.

Table 1 presents the percentages of the different types of answers according to age/grade.

Table 1: Percentages of Different Types of Answers to Question 1 (the lottery game)

	Grades			
Answers	5	7	9	11
Vered has a better chance	0	0	0	0
Ruth has a better chance (main misconception)	70	55	45	35
The chances are equal (correct answer)	30	45	55	65

As one can see, the answers improve with age. One deals here with what Tversky and Kahneman called the representativeness misconception: People tend to estimate the likelihood of an event by taking into account how well it represents the parent population. Table 1 shows that its influence decreases with age.

The Negative and Positive Recency Effects (See Cohen, 1957; Fischbein, 1975, pp. 51-62) Question 2

"When tossing a coin, one gets one of two possible outcomes: Either heads or tails.

Ronni flipped a coin three times and in all three cases "heads" came up. Ronni intends to flip the coin again. Is the chance to again get "heads" the fourth time, smaller, equal, or bigger compared with the chance of getting "tails"?

The correct answer is that "heads" and "tails" have the same chance to come up with the fourth launching.

In Table 2 we present the results, in percentages, according to the different types of answers.

Table 2: The Negative and Positive Recency Effects. Categories of Answers (in percentages)

	Grades			
Answers	5	7	9	11
The chance of getting "heads" the fourth time is smaller (the negative recency effect - the main misconception)	35	35	20	10
The chances are equal (correct answer)	40	55	70	90
The chances of getting "heads" is bigger (the positive recency effect)	0	5	0	0
Other types of answers	25	5	10	0

The impact of the misconception expressing "the negative recency effect" is obviously decreasing with age and the frequencies of correct answers ("the chances of getting 'heads' on the fourth launching are equal") increase with age. The positive recency effect is almost absent.

Question 3 (See Tversky & Kahneman, 1982, pp. 4)

In a certain town, there are two hospitals, a small one in which there are, on the average, about 15 births a day and a big one in which there are, on the average, about 45 births a day.

The likelihood of giving birth to a boy is about 50%. (Nevertheless, there were days in which more than 50% of babies born were boys and there were days in which less than 50% of babies born were boys.)

In the small hospital one has kept a record during a year of the days in which the number of boys born was greater than 9, which represents more than 60% of the total of births in the respective hospital.

In the big hospital, one has kept a record during a year, of the days in which there were born more than 27 boys which represented more than 60% of the births.

In which of the two hospitals there were more such days? Choose and encircle the correct answer.

A. In the big hospital, more days were recorded where more than 60% boys were born.

B. In the small hospital, more days were recorded where more than 60% boys were born.

C. The number of days where more than 60% boys were born was equal in the two hospitals.

Explain your answer.

The text of this question, as presented in the original version (Kahneman & Tversky, 1982, p. 44), was slightly modified here in order to make it more accessible.

The results are presented in Table 3.

Table 3: Question 3 - The Effect of the Sample Size (in percentages)

	Grades			
Answers	5	7	9	11
A. In the big hospital, more days were recorded where more than 60% boys were born.	20	35	5	10
B. In the small hospital, more days were recorded where more than 60% boys were born.	0	0	5	0
C. In the two hospitals the same number of days was recorded (main misconception).	10	30	70	80
Other answers	10	5	5	10
No answer	60	30	15	0

The basic misconception, in this case, is that the number of days in a year in which one has recorded the birth of more than 60% boys does not depend on the sample size (Kahneman & Tversky, 1982, p. 44).

As one can see by inspecting Table 3, this misconception increases with age in a surprisingly regular manner. Starting from 10% in grade five, it reaches about 80% in grade 11. The erroneous answer ("The same number of days in which more than 60% of boys were born") was usually justified by the equality of rations: "9/15 = 27/45 - they express the same ratio".

The Effect of the Time Axis ("The Falk Phenomenon", see Falk, 1979)

Question 4

Our wording of the text of question 4 is, somehow, different from the original, in order to avoid the term probability and, especially, the requirement to calculate probabilities. Let us remind that the subjects did not possess any formal knowledge in the domain of probabilities.

Yoav and Galit receive each a box containing two white marbles and two black marbles.

A. Yoav extracts a marble from his box and finds out that it is a white one. Without replacing the first extracted marble, he extracts a second marble. The likelihood that this second marble is also white is smaller, equal or bigger compared to the likelihood that it is a black marble?

Explain your answer.

This is a relatively easy question. After the extraction of a first white marble, two black marbles and one white marble remain in the box. Consequently, at the second extraction, the chances of getting a white marble are smaller than those to get a black marble.

B. Galit extracts a first marble from her box and puts it aside without looking at it. She then extracts a second marble and sees that it is white.

Is the likelihood that the first extracted marble is white, smaller than, equal to, or bigger than the likelihood that it is black?

At a first glance, the question seems easy: at the first extraction the chances are equal.

In reality, things are different: knowing that, at the second extraction, one has got a white marble, changes the situation. The chances are higher that the first extracted marble is black, exactly as in the first problem.

We have divided the answers of the subjects into three categories:

Category 1:

A: The subjects who answered to the first question that the likelihood of getting a white marble (the second extraction) is smaller than the likelihood of getting a black marble (correct) and B: The same subjects answered that the chances that the first marble (first extraction) would be white is also smaller (correct, as above).

Category 2:

The subjects, who answered to question A, that the chances of getting a white marble is smaller (after extracting a white marble) and to question B: that the chances that the first extracted marble is white are equal to the chances that the first extracted marble is black (incorrect answer - the expected misconception).

Category 3:

Subjects who answered to both questions (A and B) that the likelihood of a white marble is equal to that of a black marble (both incorrect answers).

In Table 4, one may see the percentages of answers distributed according to the three categories, and their evolution with age.

Let us consider category 2 (that is, the main misconception). At the first extraction there was the same likelihood to extract a white or a black marble, while, at the second extraction, the likelihood to extract a white marble is higher (after extracting first a white marble). We got the following percentages: 5%, 30%, 35%, and 70%.

We have here another example of an intuitively based misconception, the frequency of which grows with age.

Table 4: Question 4 - The Axis of Time (in percentages)

	Grades			
Answers	5	7	9	11
Category 1 (both correct)	45	50	35	30
Category 2 (the first correct, the second incorrect) (the main misconception)	5	30	35	70
Category 3 (both incorrect: equality of chances for both questions)	25	15	25	0
Others	25	5	5	0

In the present problem, it is the general principle of causality, with its apparent one-directedness which seems to become more and more effective with age in structuring the intuitive interpretation: what happens at the second extraction cannot influence, retroactively, what has already happened at the first extraction! What the subjects do not seem to realize is that the knowledge of the second outcome should participate in determining the probability of the first outcome. The problem is not of a physical nature in which the antecedent determines the consequence, but of a pure logical nature. This more subtle level of rationalization is not reached by many subjects.

The Interpretation of the Findings

A plausible interpretation of the above findings could be the following:

In each intuition considered, a general intellectual schema is embedded which influences the conclusion. The schema acts tacitly and this implies, in our opinion, that the schema becomes an integral part of the respective intuition. But, at request (when a justification is required), the schema may be rendered explicit by the subject.

Let us mention those principles which could be identified in the structure of the intuitions considered above.

- The problems related to representativeness (the negative recency effect and the higher likelihood of a group of random numbers in winning in a lottery game). The basic idea is the independence of outcomes in a stochastic experience. It is this principle, this intellectual schema, which improves with age and, finally, overcomes the primitive, global, intuitive heuristic of representativeness.
- A second schema, identifiable in the two hospitals' problem is related to the equivalence of ratios. No matter the size of the samples and the numbers involved, one may have the same ratio $\frac{a}{b} = \frac{c}{d}$ despite the fact that c is different from a and d is different from b. The child and later on the adolescent, learn gradually, the concept of equivalence of ratios, which lead to the schema of proportion.

The concept of ratio is involved in the solution of the problem of the two hospitals. But, it is incorrectly involved! Apparently one deals with ratios in the attempt to solve the respective problem. This belief is misleading. Instead, one has to consider another stochastic law: the law of large numbers. As the sample size (or the number of trials) increases, the relative frequencies tend to come closer to the theoretical probability. And, on the contrary, if one considers a small sample, the relative frequencies of expected outcomes may deviate largely from the theoretical probability.

Let us now consider another misconception, the impact of which, as it has been found, becomes more effective with age. We refer to the "Falk phenomenon": The second extraction of the ball cannot influence the previous one. In this case too, it is a general principle, deeply rooted in our mental activity, which determines the answer - the principle of causality: the antecedent determines the consequent. In this strongly intuitive view, it is the time axis with its one-directedness which dominates the reasoning. An effect cannot act retroactively. It is this schema which is decisive for the respective intuition. Because of its strength, the subject neglects an essential information: one knows that the second marble extracted is white. The most striking, direct, influential element, is the empirical principle of causality and this determines the answer. The idea that one has to shift from a concrete, causal, time-oriented relation, to a formal mathematical one, is much more subtle. With age - as our findings have shown - the subject becomes more inclined to employ, in his reasoning, the causality principle and the irreversibility of time - despite the fact that they are not adequate in solving the respective problem.

The general idea is then, that intuitions are usually manipulated from inside, tacitly, by certain basic intellectual schemata. Naturally these schemata develop with age. In certain circumstances, they may be adequate to the constraints of the problem (for instance, the independence of outcomes in stochastic processes) and those results in the improvement with age, of the respective intuitions. In other circumstances, the respective schema, though formally correct in itself and improving with age, does not really fit the problem and one gets worse results as an effect of age!

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The Fennema-Sherman 'Mathematics as a male domain' scale: A Re-examination

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Abstract

The contributions made by the Fennema-Sherman 'Mathematics Attitude Scales' [MAS] to research on gender differences in mathematics learning outcomes cannot be underestimated. The construct 'mathematics as a male domain' remains a critical variable in explorations of the continued disadvantage experienced by females in the field of mathematics. However, recent research evidence suggests that several items in the 'Mathematics as a male domain' scale of the MAS are no longer valid. Findings from four separate research studies are presented that challenge the interpretation of particular responses to these items. The scale requires revision if it is to measure accurately its originally operationalised construct.

Introduction

Mathematics education researchers interested in gender issues have used the Fennema-Sherman [F-S] *Mathematics Attitudes Scales* [MAS] (Fennema & Sherman, 1976) extensively to measure attitudes towards mathematics. Fennema and Sherman's (1977) first study to report findings from the MAS is among the most cited articles in mainstream journals of educational psychology (Walberg & Haertel, 1992). In that article, Fennema and Sherman (1977) discussed their findings from the *Mathematics as a male domain scale* [MD]:

> Male responses differed significantly from female responses. While boys did not stereotype mathematics strongly as a male domain on this scale, they always stereotyped it more strongly than did females. (p.68)

A meta-analysis of mathematics education research studies incorporating affective variables examined by gender was conducted by Hyde et al. (1990). Among the variables examined, 'mathematics as a male domain' was identified as having the greatest gender difference (largest effect size). The results showed that gender differences had declined during the 1980s, but were still large. For females' potential in mathematics, this was of concern:

> ...the stereotyping of math as a male domain may be critical to females' willingness to achieve in mathematics. It may indicate a pervasive belief throughout much of society, which females sense and find difficult to overcome. (Hyde et al., 1990, p.312)

The MAS were published in 1976. The assumptions underpinning the development of the MD were noted by Fennema and Sherman (1976, p.7):

> The less a person stereotyped mathematics, the higher the score. This is done to fulfill the purpose of the scale development as it was assumed that the less a female stereotyped mathematics as a male domain, the more apt she would be to study and learn mathematics.

Presumably, the corollary of this assumption is that low-scoring females would believe mathematics to be a male domain and would thus be less likely to study and learn mathematics. Evidence suggests that low scores on the MD can no longer be interpreted as necessarily reflecting the stereotyping of mathematics as a *male domain*. When the scale was developed, there was no apparent allowance for beliefs that mathematics might be considered a *female* domain. This view seems consistent with prevailing Western societal views of the 1970s.

In this paper we provide evidence from four separate research studies implying that the MD needs revision. A brief overview of the criteria relevant to the exemplary design of psychometric scales precedes the presentation and discussion of the research findings.

Criteria for a well-designed psychometric scale

The MD is an example of a summated-ratings scale with each of its 12 items (six positively and six negatively worded) scored on a 5-point Likert-type scale. Criteria for evaluating the quality of such scales have been well-established in the psychometric literature. A soundly constructed procedure should include the following steps:

- defining as clearly as possible the construct under investigation;
- ensuring that all the items in the scale display content validity, are related to the intended construct, and are monotonic;
- checking that all items are free of technical faults;
- selecting a set of responses which are consistent with the wording of the item, and ensuring that responses are scored correctly;
- conducting a trial of the scale in order to ascertain item and scale sensitivity, item and scale internal consistency, and whether the scale displays unidimensionality, convergent validity, discriminant validity, stability and predictive validity.

The description of the development of the MAS (Fennema & Sherman, 1976) indicates that consideration was given to these points. Findings from the research studies presented in the next section suggest that if these tests were conducted today some of the items on the MD might well be rejected.

The four research studies

In this section, brief descriptions of four separate research studies, and relevant findings that challenge aspects of the MD, are presented. Participants were drawn from different countries and age groups.

Studies 1 and 2

In separate studies, Grade 9 students in two countries, Australia and Sweden[1], were asked if men or women were better at mathematics and to give their reasons. The questions were posed in an open-response format and were part of a larger self-report instrument. The sample size in Australia was 187 (97M, 90F). In Sweden there were 76 students (34M, 41F, 1?). Responses were sorted into categories and percentage frequencies for male and female students in each country are shown in Table 1. Typical examples of the reasons given are shown in Table 2.

Table 1: Percentage frequencies of responses for male and female students in Australia and Sweden.

		Men better	Women better	Equal	Unsure	Other
Australia	Males	23	13	45	13	6
	Females	2	9	77	2	10
Sweden	Males	7	27	44	12	10
	Females	21	41	15	15	8

Table 2: Typical responses from Australian and Swedish students

Australia	Sweden
Men, because in a co-ed class, boys would seem to be better than girls because they dominate (Male)	*Men*, because they are probably smarter in this area. (Female)
Women, because they are more mature and take it more seriously (Female)	*Women*. Because they are smarter and learn quickly. (Male)
Equal, because in the past women have not been allowed to be professors and are under more pressure (Female)	*No difference*. It is individual. (Male)

We thank our Swedish colleagues, Barbro Grevholm and Kerstin Persson, for their help with translations and in the administration of the survey.

The data in Table 1 clearly show that some students in both countries consider women to be better at mathematics than men. This trend was more marked in Sweden than in Australia. From the Swedish data, it might be argued that many students now perceive mathematics to be a 'female' domain. Some of the reasons given by students from both countries reflected cognisance of societal and classroom factors that have constrained women's mathematical attainments in the past and that might still persist today (see Table 2). Some explanations revealed stereotyped expectations (e.g., men smarter, females have to work hard to succeed), while others reversed these patterns.

Study 3

Conducted during the early 1990s with Australian grade 7 mathematics students, this study aimed to examine the relationships between a set of affective variables included in explanations for gender differences in mathematics learning and classroom factors. 'Mathematics as a male domain' was one of the affective variables investigated. The study was conducted in two parts. Large scale survey data (N=782: 386M, 396F) were complemented by in-depth studies of two mathematics classrooms in which two male and two female students had been targeted for closer study.

An adapted 6-item (three negatively and three positively worded) version of the MD was used in the survey (score range: 6-30). A large gender difference was found ($p<.001$, effect size=.68) with the females' mean score (27.09) higher than males' (24.72). Thus females were less likely than males to perceive mathematics as a male domain. A reliability check revealed a Cronbach alpha value of .67 for the scale, considerably lower than the published Fennema and Sherman (1976) split-half reliability value of .87. This alpha value may have been sample specific. With societal changes over time, response patterns to certain items may also have changed, rendering some previously sound items 'faulty'.

The in-depth classroom studies suggested that it may be inaccurate to infer from high MD scores that students believe mathematics is not a male domain. Classroom observations implied that students might hold more stereotyped beliefs, that such beliefs might remain unchallenged, or may be fostered in the classroom. Findings from one of the mathematics classes and the four students (2 M and 2 F) targeted for closer study are described briefly to illustrate these points.

The class mean score (28.00) on the MD was much higher than was found in the large scale survey (25.93), implying less stereotyping of mathematics as a male domain by this class than was the norm. The classroom climate was a potent factor associated with stereotyping, however. It was noisy, male horseplay was tolerated, girls misbehaving similarly to boys received sterner disciplinary action, and more learning tasks were set in contexts that would appeal to boys than to girls. The teacher claimed that he was aware of gender issues. He believed he had made efforts towards redressing known inequities. There was little evidence to support his feeling that he may have swung too far "to the detriment of the boys in the class, perhaps".

The two girls' scores (Cara[2]: 29 and Jill: 30) on the MD were higher than the two boys' (Ron: 28 and Stan: 22). Both girls' scores exceeded the female cohort mean (28.38), Ron's score was slightly higher than the male cohort mean (27.55) and Stan's was well below. The scores suggested that Cara, Jill and Ron did not believe that mathematics was a male domain. From his classroom behaviour and other evidence, this did not appear to be true for Ron. Over four of the 15 sequential lessons observed, the targeted students worked together on a cooperative small group task based on the word game Scrabble (arguably a 'female-oriented' context). Compared with other tasks with male-oriented contexts (e.g., football, basketball and horseracing), both Ron and Stan clearly disliked the task. For the four lessons, they were totally uncooperative, used sexist language, taunted the girls, and avoided doing much work, particularly tasks they appeared to consider gender-role appropriate for girls (e.g., ruling up and colouring).

Among other self-report data gathered, the students were asked whether men or women were better at mathematics (same question as for Studies 1 and 2 above). Ron replied that it was 'sexist' to say whether men or women were better. Stan wrote that men and women were equally capable, which was clearly inconsistent with his low MD score. From the boys' answers it was inferred that there was an awareness of the socially acceptable responses expected about this issue.

The findings from the survey suggest that changes in response patterns to the MD may have occurred in the two decades since in was developed. The in-depth studies challenge the reliance that might be placed on interpretations of high MD scores (less stereotyped people).

Study 4

In June 1994 a 35 item questionnaire on gender role orientation was posted on five electronic mailing lists with themes on the social context of learning; gender, science and technology; and teacher research. Replies were received within two weeks from 34 males and 125 females from over 20 different countries. One of the items, *Women are certainly logical enough to do well in mathematics*, was taken from the MD. Most strongly agreed (80%) or agreed (15%) with it. The *E-mail* format of administration was used to encourage respondents to elaborate on their rating. Representative comments are shown in Table 3.

The responses indicate a rejection of the assumption that males are more logical than females and further suggest that a '(Strongly) disagree' rating for this item may indicate disagreement with the assertion that *logic* is an important requirement for success in mathematics. Thus the item now appears to tap beliefs about two constructs: mathematics as a male domain - but with females now often asserted to be more logical, or about the relevance of logic to mathematics - with logic not being believed as critical for success in mathematics as implied.

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[2] Fictitious names have been used for the students

Table 3: Representative elaborations

Women are (more) logical	Logic not needed to do well in mathematics	Cannot generalise
Your question was ill-phrased. More of the women than the men I know are good at math	Always assuming that "doing well" is dependent on logic	Some women are, some women aren't; some men are, some men aren't
Ask any woman who runs a house, works a job and juggles her and kids schedules, and gets her bills paid on time	But is math really "logical"? Mostly it is memorization and use of rules	A few women or all women?
More logical in fact	Math takes intuition as well as logic, don't forget!	Some are, some aren't - just like men

Discussion and implications for the F-S "mathematics as a male domain scale

The findings from studies 1 and 2 challenge the assumption underlying the interpretation of low scores on the MD. Since it is now evident that some people consider females to be better at mathematics than males, low scorers may not necessarily stereotype mathematics as a 'male' domain, but might consider mathematics to be a 'female' domain. Interpretations of responses to the sample item from the MD illustrate this point. Consider:

> Studying mathematics is just as appropriate for women as for men
> (Study 3: Studying mathematics is equally important for women and men)

Students who agree with the statement do not stereotype mathematics as a male domain. According to the underlying assumption of the scale, those who disagree believe mathematics is more appropriate (important) for men. Disagreement may, however, reflect beliefs that mathematics is more appropriate for women. It is the assumed monotonicity of this original scale item, and others like *'Girls can do just as well as boys in mathematics'*, that no longer holds up. *Monotonic items* are those for which there is a steady rise (or fall) in the probability of endorsement as one moves across the attitude continuum. Non-monotonic items are out of place in a summated ratings scale, where a high score ought to indicate a large amount of the variable being measured. For the MD, this should be a large (or small) amount of 'mathematics as a *male* domain'. It would appear that this is no longer necessarily the case.

The Cronbach alpha value reported in study 3 was moderate and considerably lower than the reliability value published with the MAS (.67 and .87 respectively). Alpha is a measure of the internal consistency of a scale. A scale will display internal consistency (i.e., alpha value) if each of its items correlates moderately with some (but not necessarily all) other items in the scale. While needed, a high alpha value alone is an insufficient condition for scale unidimensionality (all items measuring the same construct) (see Green, Lissitz & Mulaik, 1977; Gardner, in press). Factor analysis is a preferred method to establish dimensionality. In addition, the *content validity* of items entails more than a simple check that wording is related to the construct of interest. For each item, it involves considering if the responses provide useful information about the construct.

For the class described in Study 3 and for some of the individual students, the classroom behaviours observed were inconsistent with the level of stereotyping that could be inferred from MD scores. In particular, the findings for Ron challenged the reliance that might be placed on the interpretation of high scores (less stereotyping). The data for Ron and Stan suggested that 'political correctness' may partially account for some of the inconsistencies with MD scores. Cook and Selltiz (1970/1964) cautioned that distortions, in directions of 'social desirability', were possible on self-report measures of constructs for which the implications of responses were apparent to the respondent. Given that the level of awareness of gender-related issues in Western societies has increased since the time of the development of the MD, 'politically correct' responses may today skew upwards MD scores. Considering the high means on the MD obtained in Study 3, scale items may no longer display acceptable degrees of *sensitivity*, a diversity of responses in a sample of respondents. Upwardly skewed scores would also present an inaccurate impression of change in the stereotyping of mathematics as a male domain over time, as reported in meta-analyses, for example.

From the item, *'Women are certainly logical enough to do well in mathematics'*, used in Study 4, it is apparent that some respondents challenge the assertion that logic is needed for success in mathematics and that their answer focuses on this aspect rather than on presumed gender characteristics. There is thus an unacceptable element of ambiguity about the response given to this item.

Implications for the MD and recommendations

'Mathematics as a male domain' remains a relevant variable in explanations for persistent patterns of gender difference in some mathematics learning outcomes There is no compelling evidence to refute this claim. On the contrary, evidence suggests that the extent to which mathematics continues to be stereotyped as a male domain may have been underestimated, from MD scores for example. The construct was clearly defined when the MAS were developed. However, two decades later a re-examination of the construct, from a contemporary perspective, is required.

From the evidence presented from the four studies discussed above, other revisions to the MD are called for. Recommendations are as follows:

- the direction of scoring on the scale should be reversed from that used in the original so that a high score will represent a large amount of the variable of interest, "mathematics as a male domain"
- Items on the present scale should be carefully re-examined for content validity and for monotonicity. Items that do not conform should be rejected
- New items should be written and tested widely. Consideration should be given to all the criteria for a well-designed scale noted earlier
- A new scale should be published and endorsed by the original authors

Access to well-designed, valid and reliable psychometric scales is invaluable for the researcher committed to a particular research field. Till now, the widely used MAS have been very effective in fulfilling this role for those investigating gender issues in mathematics education. One of the MAS scales, the MD, is sorely in need of revision. Otherwise, those using it may unknowingly obtain inaccurate measures of the construct "mathematics as a male domain", with the inevitable consequences on data interpretation.

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STUDENTS' MISCONCEPTIONS WITH DECIMAL NUMBERS - PRELIMINARY RESULTS FROM A STUDY OF COMPUTER BASED TEACHING

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Abstract

A year long study of 242 Norwegian students aged 10 - 14 investigated the development of their understanding and performance with decimal numbers using computers to support a constructivist teaching approach. Spreadsheet tasks were used to stimulate conflict and discussion. A further 394 students served as a control group. Pretests, posttests and delayed posttests were used for quantitative evaluation. The computer group performed significantly better with the largest effects from high spreadsheet users. An item analysis of pretest data revealed a pattern of misconceptions which supports and elaborates those reported in other studies. Additional data were obtained from classroom observations, interviews and questionnaires. This preliminary report gives only a partial picture of the study.

Introduction

The use of computers in mathematics teaching have a huge potential for improving students performance and understanding in mathematics. This is argued in several articles and reports both as a common sense opinion and on basis of some research. (Bennett, 1991; Watson, 1993)

Looking at computer software, we may be concerned about what kind of learning theory lies behind the way the software presents tasks to the students. A lot of small pieces of software designed to teach number skills can be classified as drill and practice software. The behaviourist thinking of stimulus and response seems to have a lot of support in this kind of software. Could computers also be used to stimulate a constructivist teaching approach, using computers to stimulate discussion and problem solving activities?

The aim of the research was to study development of students' understanding and performance in decimal numbers using the computer as a support with emphasis on a diagnostic teaching approach.

Previous research in this area

There exists a substantial body of research concerning students' misconceptions of decimal numbers. Problems are connected to the meaning of decimals and place value and shows up in different tasks such as ordering of decimal numbers, density of the number line, addition and subtraction using decimal, multiplication and division. (Swan, 1983) The concepts have to be adjusted or accommodated to the new number concept. Application of previous knowledge from natural numbers to decimals does not give the correct understanding. (Resnick, Leonard, Omanson, & Peled, 1989; Bell, Fischbein, & Greer, 1984) The students need new experience to explore and discover the new properties of the number concept by introducing decimal numbers.

The diagnostic teaching method has been explored in several studies, and has also been used in a teaching experiment on decimal numbers where classes using calculators in a conflict and discussion method were compared with classes using a positive only approach. (Bell, 1993) The results show a clear preference to the diagnostic teaching method.

Research has also been done using a spreadsheet in mathematics teaching in different contexts. The concept of variable and understanding of elementary algebra have been explored in various projects. (Rojano & Sutherland, 1991; Rojano, 1993; Beare, 1993)

Research methodology

The research classes used computers through a whole school year, in different areas of the mathematics. The computer use was not confined to the topic area of decimal numbers. They used different kinds of software, small programs designed for teaching number skills, some games for problem solving and tasks to solve on a spreadsheet.

The research and control classes were given a test in decimal numbers, a pre test in the beginning of the school year, a post test in the end of the school year and a delayed post test after the summer vacation. I also visited the research classes, observed several lessons when students used computers and interviewed students and teachers.

There were two sets of control classes, some not using computers at all, and some using computers but not using the spreadsheet tasks prepared for the research and having no information about what software the research classes used. But all teachers, both in research classes and in control classes were given information about typical misconceptions in decimal numbers after the pre test. The research was done with students in the year levels 5, 6 and 7, that is of age about 10 to 14 starting school at the age of seven. The sample consisted of 242 students in research classes, 297 students in control classes not using computers and 97 in control classes using computers.

Preparation and treatment

Computers have been used in Norwegian schools for many years, but for this age level mainly in special needs education, and rarely in ordinary mathematics classes. Only a few teachers in the research classes had some experience using computers in their classes before. As a preparation I gave an introduction course into the use of computers in mathematics teaching, an introduction to spreadsheets and to the use of the software and worksheets that I planned for use in the teaching project. In this way they all had the same information about the idea of the teaching of decimal numbers using computers and we discussed how to organise and plan the computer use in their mathematics teaching. The teachers were not given a strict plan to follow, but had to plan their teaching and use of software in their classes. However, I did ask them all to use the worksheets with spreadsheet tasks planned for the teaching of decimal numbers. My intention was that the implementation in the classes should be the teachers responsibility, and the computers be a natural part of the mathematics classroom. In this way the treatment implemented in the research classes would be a realistic model for other ordinary teachers.

Results from the pre test

The test in decimal numbers consisted of 74 single items covering a broad area of decimal numbers: the concepts of place value, ordering numbers, density on number line, reading scale and calculations involving both pure number work and use in a practical context. The results from the pre test in decimal numbers revealed misconceptions similar to what is seen from other research. I will give a few examples from the first part of the test to illustrate this. I follow the Norwegian convention of using a decimal comma instead of decimal point.

Task 5 dealt with the problem of number size:

> 5.
>
> a Draw a ring around the biggest of the three numbers.
>
> 0,62 0,236 0,4
>
> b How do you know it is the biggest? ________

The most common error was to give 0,236 as the biggest number instead of 0,62. In year 5 60% of the students gave this answer, and only 31% answered correctly. In year 6 and 7 this wrong answer was given by 42% and 19% and correct answers 50% and 77% of the students. In year 5 only 17% could give a correct reason for their answer in item 5b, 27% gave the explanation "most digits behind the comma" and 51% gave other answers or no answer at all. In year 6 and 7 more students gave a correct explanation, 33% and 60% in year 6 and 7.

In task 6 the students had to write four given decimal numbers in increasing order. The first task ordering 0,3 0,7 0,6 0,1 was done correctly by most students, more than 90% correct in all year groups. This task however may be performed correctly also by students having a misconception of the decimal numbers. But ordering numbers with different number of decimal places revealed difficulties. In the task of ordering 0,62 0,25 0,5 and 0,375 increasingly, correct answers were given by 20%, 35% and 62% and the most common error: 0,5 0,25 0,62 0,375 was given by 56%, 41% and 21% in the year levels 5, 6 and 7 respectively.

Students seem to think of a the digits behind the comma as a separate number and compare without taking account of the decimal comma. Looking at the decimal number as a number pair, seems to be a common misconception.

A few tasks involving adding also showed problems of place value:

11. Write the next two numbers in each sequence:

a	0,3	0,6	0,9	____	____
b	0,92	0,94	0,96	____	____
c	1,13	1,12	1,11	____	____

At first the results from the three items in Task 11 seemed a bit surprising. Task 11a appeared to be more difficult than the two next, with 35% 36% and 40% of the students giving the most typical wrong answer of 0,12 and 0,15 as the next two numbers, and only 17%, 34% and 35% (in the years 5, 6 and 7 respectively) answering correctly. Task 11a looks easy and the next two more difficult. A possible explanation could be that the students can easily count in threes giving 3, 6, 9, 12, 15 and so on. Therefore the result here also by pure routine becomes 0,12 and 0,15 for the next two numbers. Looking at the next two items, the routine for counting is weaker and therefore the students have to concentrate more, and get it right. This explanation was also confirmed later by looking at students working on this kind of tasks. When prompted to have a closer look at the task, the students discovered the error and corrected it.

13. Add 0,1 to:

a	4,256	Answer: ________
b	6,98	Answer: ________

14. Take away 0,1 from:

a	15,863	Answer: ________
b	1,06	Answer: ________

In task 13 the students had to add 0,1 to the numbers 4,256 and 6,98. The most common error in both items in this task was to add one to the last digit behind the comma giving the answer 4,257 instead of 4,356 in 13a and the answer 6,99 instead of 7,08 in 13b. In year 5 these errors were done by 40% and 44% of the students and correct answers given by 40% and 24%. In year 6 and 7 the results were better, but still in year 7 19% of the students gave the wrong answer 4,257 to the task 13a and 21% gave the answer 6,99 to 13b.

Similar results showed up taking away 0,1 from the numbers 15,863 and 1,06. In year 5 the fraction of correct answers were 35% to task 14a and 18% to task 14b. Students gave the wrong answers of 15,862 and 1,05 most frequently, 41% and 47% in year 5 and 18% and 26% in year 7.

The most frequent wrong answers to the tasks 13 and 14 followed the same pattern of adding or subtracting from the last digit behind the comma regardless of place value. Looking at cross tables of pairs of the items 13a, 13b, 14a and 14b, I found that in all combinations about 30% of the students made this particular error to pair of these tasks. I also found that 35% of the students in year 5 made this kind of mistake in all the four items, 13a - 14b. In year 6 and year 7, 28% and 16% did the same mistake.

The fraction of students answering correctly was from 28% to 51% to pairs of these tasks. This supports the idea of a consistent pattern in the students thinking.

How can computers help?

Computers may be used in different ways in the classrooms to support different teaching styles. Looking at small pieces of software designed for mathematics teaching, I found that a drill and practice approach was the most common. But the computer can also be used as a support for a diagnostic teaching method, creating conflict and discussions amongst the students, and in this way help them develop concepts and understanding of the decimal numbers. Some worksheets were prepared with this teaching style in mind.

In Worksheet 5 the students explored number sequences using a spreadsheet. By putting in a starting number, a formula in the cell below and then copy the formula further down they may generate number sequences as shown in this extract from the worksheet:

	A	B	C	D
1	1	0,3		
2	=A1+1			
3	=A2+1			
4				

a Write the number 1 in cell **A1** and the formula =A1 + 1 in **A2**. The next formula in **A3** is going to be A2 + 1. Copy the last formula down. Look at the result. Then put another number in A1 and look at the result.

Then put the number 0,3 in **B1** and write a formula in **B2** so that the number in that cell is 0,5. Copy the formula down and study the result.

b Study these number sequences and write in the next four numbers in each column:

0,1	0,2	0,01	0,12	7,6	1,17
0,2	0,4	0,03	0,135	6,3	1,15
0,3	0,6	0,05	0,15	5,0	1,13
0,4					

Make the same number sequences on a spreadsheet and compare your results.

Similar ideas were used to look at multiplication and division involving decimal numbers. Some other worksheets were also prepared for other number experiments and using decimal numbers in a practical context, e.g. a shopping list or price labels.

In a class of year 7 the students had been working on a spreadsheet in a few lessons before, making formulas and copying on a spreadsheet to make number sequences. The students, starting on Worksheet 5, at first worked away from the machines, writing the next numbers in the first three or four sequences and a suggested formula to make the number sequence. Then they turned to the computers to make the same sequences and compare results. In this way they discovered mistakes and explored further connections between numbers. I observed several groups of students doing similar mistakes, the same error pattern as in the Task 11 in the pre test.

Lisa and Mary did some quite common errors when they were writing numbers further down in column D and E

C	D	E
0,1	0,2	0,01
0,2	0,4	0,03
0,3	0,6	0,05
0,4	***0,8***	***0,07***
0,5	***0,10***	***0,09***
0,6	***0,12***	***0,011***
0,7	***0,14***	***0,013***
0,8	***0,16***	***0,015***

Fig. 1 Lisa and Mary's work

(students writing in italics and bold)

They gave the correct formula i D2, = D1 + 0,2 but wrote =E1 + 0,01 for the sequence in the E column. Then, at the computers they discovered their errors and corrected them. I observed what they were doing, and asked why this was wrong, referring to their numbers 0,6, 0,8, 0,10. *"Oh yes, 0,10 is less than 0,8 and it has to be bigger."* the students replied.

Lisa seemed to be convinced, but Mary still seemed not quite happy about this.

The teacher arrived, and asked - *"is 0,8 less than 0,10?"* - to help them conclude.

These student dropped the next problem starting with the numbers: 0,12, 0,135, 0,15. Probably they found it too difficult. But they did right on the next two sequences which were: 7,6, 6,3, 5,0, .. and the sequence starting: 1,17, 1,15, 1,13 They seem to have learned from their first mistake.

Two boys, Tor and Lars had problems with the sequence starting 0,12 0,135 ... They tried to add 122, but then - *"it should be nought comma ... something"* they said. They tried to add 0,112 and they tried 0,12 but it did not work. Later Tor found he could add 0,015 but his partner still felt not sure about this.

I also observed a student in year 5 working on the same problem. Peter had easily made the first few number sequences in Worksheet 5, then looking at this one.

After a while he came up with the number 0,165 on his paper. But then he said to himself *"... must divide here, it is going down.."* He was reading nought comma hundred and thirty-five and nought comma fifteen. I (AB) observed him, heard his comments, and tried to give hint, but not to solve the problem for him:

0,12
0,135
0,15
0.....

AB: *Will it be less? Is 0,135 less than 0,15?*
P: *Oh, no ....* (he withdraw his argument)
P: *But it must be 15, must add 15*
AB: *Do you mean 15?*
P: *No, I mean 0,15* (saying nought comma fifteen).

He tried, but it came out wrong on the spreadsheet compared to the given sequence and he could not see why. Peter: *Then it has to be 0,115* ... He tried, looked at the result. "*Oh, no this is too much also* ... "

Peter struggled a lot to find the connection between the decimal numbers, but he had no problem trying out his suggestions by writing a formula and copying to make the number sequence on the spreadsheet.

Working on this worksheet, the students had a lot of discussion, quite lively, as they were working comparing the computer results with their own written number sequences. Several groups made mistakes when writing sequences on paper and in this way got the contradictions to look at when they tried to make the same on the spreadsheet. They accepted the answers given by the computer and seemed to arrive at an agreement on how to correct their mistakes. The use of the spreadsheet and copying formulas was not the problem, but the mathematical content in the tasks was time consuming.

Comparing research and control groups.

A first look at the results from the pre and the post tests in decimal numbers showed that the research classes improved only slightly more than the control groups. Both the research and the control groups improved also from the post test to the delayed post test.

From observations in the classes and from the teachers reports, I know that all the classes in the research group used computers regularly during the year. But still, there were differences in the way they used computers and especially to what extent they used spreadsheets and the tasks designed to teach about decimal numbers and other small pieces of software. I therefore split the research group into two parts in each year level, the "High" group were high users of spreadsheets whilst the "Low" group gave less emphasis to spreadsheet tasks.

Comparing the different groups in the research I found that the "High" group generally improved more than both the control groups and the "Low" group in each year level. To test for significant differences between the groups I used analysis of variance. The differences were significant in year 5 and 6 and partly in year 7. This effect was found mainly to be due the fist part of the test which contains the tasks on size, ordering, adding and subtracting decimal numbers, e.g. the tasks 5, 6 11, 13 and 14.

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Conclusions

From the observations I found that the spreadsheet tasks engaged the students and stimulated discussions. The students had only small problems in using the computers, and could concentrate on the discussion of the mathematics. The strongest test results in favour of the research group "High" was found in the topic area dealt with in the spreadsheet tasks that were used in these classes. More details on this will be given in a later presentation.

This was the first year of using the computers in the mathematics classroom for most of the teachers and for all the students in the research classes. Some of the worksheets prepared for the teaching of decimal numbers were not used, either because the time was limited or because the teachers felt insecure. With more teacher experience and computer use over some years, the way of using these resources could be better and probably results be improved. Further research is necessary to learn more about this.

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QUALITATIVE ANALYSIS IN THE STUDY OF NEGATIVE NUMBERS

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ABSTRACT

This article reports the results obtained by a study of negative numbers carried out with secondary school students. The historical-critical analysis and the clinical method provided the methodological components which exhibit the diverse stages of acceptance of the negative by the students. These stages lead to the definition of students profiles. Furthermore, these profiles reveal the basic content of school algebra that individuals need to achieve in order to become competent users in this subject.

INTRODUCTION

Most of the research literature on the study of negative numbers focuses on reporting the difficulties students have when confronted with operativity with these numbers. They deal with this problem by using concrete teaching models (Bell, 1982; Janvier, 1985; among others). The algorithmic mastery of the multiplicative rule of signs occupies a central role in aforementioned works. In contrast to this tendency, there are theoretical proposals where the emphasis is placed on the conceptual aspect of negative numbers and the historical evolution of this notion (Glaeser, 1981; Vergnaud, 1982; Peled, 1991). Most of this research currently deals with the partial aspects of the area of whole numbers and many of the studies analyze the concept *per se*, isolated from mathematical activity.

Our study seeks to organize and find a theoretical explanation for a large part of the existing empirical evidence on the subject in secondary education. It also attempts to determine the level of conceptualization of the negative number needed by the secondary school student in order to be able to solve the equations and problems posed in the educational programs. The study of negative numbers is dealt with in their interrelation with languages and methods of problems solving used by the student in this mathematical activity.

HISTORICAL-CRITICAL METHOD

Negative numbers are subjected to a historical-critical analysis in the context of the resolution of equations in search of elements which might explain the facts observed in the process of teaching equations, regarding the incidence of negative numbers. Chapters of old books which dealt in some way with negative numbers in the context of problems and equations were studied. Furthermore, the different levels of language were examined, as well as the methods and strategies used in problem solution. This historical period studied is basically the XIII and XV centuries, although background material from more ancient times was examined (from the Chinese, Greek, Hindu and Arab cultures). With regard to didactics, experimental work with second grade secondary school students was designed. With respect to basic content, the middle level algebra taught in schools corresponds to the historical period analyzed. In order to contrast the historical and didactical spheres, we will now describe the most relevant aspects form the historical texts we have studied. The categories of analysis which guided the research are as follows:

Language/Method/Operativity of the Negative Number / Interpretation of the Negative Number.

The following description of the texts provides evidence of the interweaving of these components.

In the *Chinese text* "Nine Chapters on the Mathematical Art" (Jîuzhâng suànshù., 250 B.C.), the translation from a rhetorical language to a language of calculus together with the method of tabulation and an operativity in the additive domain allow the movement from subtrahend to signed number[1] .

In the *Greek text* "Arithmetic, Diophantus" (III century), the rhetorical language, the use of abbreviations for the unknown and its powers, and the multiplicative rule of signs are conjugated with *the restoration of terms* (elimination of subtrahends) in the solution of equation.

In the *Hindu text* "Vijaganita, Bhâskara" (XII century), syncopated language, the method of inversion, the method of elimination of the intermediate term as well as the rules of operation for whole numbers lead to the notion of relative number. Acceptance or rejection of the negative solution depends on the context of the problem.

In the *Arab text*, Al-Báhir fil-hisáb, Al-Karaji, (X century), rhetorical language is used and the equation is formulated in positive terms (implicit geometric context). An algebra of polynomials based on the positional decimal system is found. These algorithmic processes require an operativity of signed numbers.

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[1] In this research, we identified different stages of conceptualization of negative numbers which appear both in the historical and didactical spheres. The stages are as follows: *subtrahend*, where the notion of number is subordinated to the magnitude; *signed number*, when a plus or minus sign is associated with the number; *relative number (or directed number)*, where the idea of opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continous domain; *isolated number*, where there are two levels, that of the result of an operation or as the solution to a problem or equation. Finally, the formal mathematical concept of negative number where this acquires the same status as positive number.

In "Il Flos" of *Leonardo Pisano* (XIII century), the author employs rhetorical language. In problems of a commercial type corresponding to systems of equations, he transforms the system into an equivalent one. If he obtains an equation with a negative solution, the equation is reformulated. The solution remains in the positive domain.

In the "Ars Magna" of *Girolamo Cardano* (XIV century), the author uses rhetorical language and a terminology with geometric referents for the unknown and its powers. He uses "Rules of Postulation of Negative Solutions" in problems of application. A *priori* supposition of a negative solution which, in the process of resolution, is absorved as subtrahend and a positive solution is obtained.

In "Triparty en la Science de nombres" of *Nicholas Chuquet* (1484), algebraic prose is used (rhetorical language with interspersed syncopation). He employs "The Rule of the First" in the solution of problems and equations. He possesses a complete operativity for numbers and algebraic terms. He accepts and symbolically represents negative and irrational solutions. In some cases he also accepts the null solution. He resorts to an intepretation of the negative solution based on the context of the problem.

The historical-critical analysis carried out in this work allows us to conclude that the presence of subtractive terms and the laws of the signs appear in remote times, as do the elements necessary for the operativity of signed numbers. It can be said that a crucial step in the recognition of these numbers is the acceptance of negative solutions. In Pisano's work this crucial stage begins when he deliberately introduces problems considered insoluble which lead to negative solutions. However, the lack of a symbolic language prevents him from achieving the extension of the numerical domain of solution. It will be the authors of the stage of algebraic syncopation who give meaning to negative numbers in equations and problems. Finally, we should say that the acceptance of the first negative solutions requires the conjugation of the following:

1) A syncopated language of expression; 2) a complete operativity of positive and negative number and zero; 3) an interpretation of the negative number which integrated the notions of signed number, relative number and isolated number; 4) an algebraic method (differentiation between the solution of the equation and the solution of the problem); 5) a specific context of the problem (the most representative is that of commercial transactions); and, 6) the abandon of geometric reference in the processes of solution and/or validation.

THE CLINICAL STUDY

This empirical analysis is based on the preceding historical study of negative numbers in the resolution of algebraic equations. Data was gathered in two ways: a group of 25 students at second grade of secondary school in Mexico City were asked to answer a questionnaire; individual clinical interviews were recorded by video and used to select a group of 15 students for clinical observation. The issues dealt with in the interview were: 1) numerical operativity; 2) resolution of linear equations; and, 3) resolution of word problems. The results concerning aspects 1 and 2 have already been reported (Gallardo & Rojano, 1994). Here we will focus on the results corresponding to aspect 3. The dimensions of analysis were:

Method of resolution/problem solution/intepretation of solution

The methods used by the students to solve word problems were as follows[2]:

Problems of ages. Luis is 22 years old and his father 40 years old. How many years must pass for his father to be twice the age of his son?

Method of Two. (Used by four students). The student finds the problem impossible because "the 2 is always there". This refers to the difference of 2 in the units of the data given in the problem as the ages of father and son advance. Example: One student established two lists of numbers, increasing the ages starting with the ages given in the problem: 22 years, 40 years. He writes: 23, 41; 24, 42; 25, 43, and so on. He notes that the difference in the figures of the units of each pair of numbers is always 2. He concludes that the problem does not have a solution.

Method of Duplication. (Used by three students). The student arrives at the correct solution, 18 and 36, the ages of son and father respectively, but he duplicates the ages and thinks that 36 y 72 is the true solution. There were also cases where the student thought that 36 x 2 = 72 and 72 x 2 = 144, is also a solution to the problem.

Method of the Difference. (Used by four students). The student finds the difference in ages, that is 40 - 22 = 18. He deduces from this that the son is 18 years old and consequently the father is 36.

Method of Altering the Difference. (Used by two students). The difference between the ages (18) is divided in half, 9, and this value is then added to the son's age, 22. The answer to the problem thus given is 31.

Ascending/Descending Method. (Used by four students). The student increases the ages of the father and the son and finds that the problem cannot be solved. He then decides to decrease the age and arrives to the correct solution.

Algebraic Method. (Used by two students). Spontaneous formulation of the equation which solves the problem.

From the analysis of the different problems exhibited in the study, the following was concluded:

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[2] The study analyzed ten problems, one of them was selected for the purposes of this article

1. It is possible to solve the problem without expressing the solution in negative terms (method of duplication, method of difference, ascending/descending method).

2. The creation of specific methods from problem to problem occurs (6 methods).

3. The choice of the appropriate method requires the acceptance of a negative solution which is interpreted in the context of the problem (method of the difference, ascending/descending method, algebraic method).

4. When faced with problems with negative solutions the students turns to changes or adjustments in the data of the problem statement as well as the construction of sources of meaning which allows him to give plausible interpretations to the solution obtained.

5. A problem which can appear impossible to solve with arithmetical methods, is thought of as possible using algebra, once the negative solution is validated by being substituted in the corresponding equation or equations (algebraic method).

Historical-critical analysis carried out with respect to negative numbers in the resolution of algebraic equations has allowed us to establish some of the conditions which propitiate the acceptance of the negative solution of word problems by secondary school students.

STUDENTS PROFILES

The results concerning numerical operativity, resolution of linear equations and word problems led to the identification of students profiles which exhibits the conceptual level of the negative number that the students show in the different tasks presented to them. We found four distinct profiles. Following is the characterization of Profile A and Profile D[3] :

Profile A

1. Presence of multiplicative domain in additive situations. This means wrong use of the multiplicative rule of signs in additions and subtractions with integers.

2. Ignorance of the Triple Nature of Subtraction and the Triple Nature of the Minus Sign. Pupils with an advanced level of conceptualization of negative numbers recognize the triple nature of subtraction (completing, taking away and the difference between two numbers) and the triple nature of the minus sign (binary, unary and the symmetric of a number). Students belonging to *Profile* A ignore the triple nature of subtraction and the triple nature of the minus sign.

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[3] These two profiles are extremes

3. Incorrect operativity in the arithmetic and algebraic spheres. The students presented inhibitory mechanisms when faced with double signs $[-(+a), -(-a)]$[4]. The open expression of the form $x + a - b =$, $a - x - b =$, receive five different erroneous interpretations: i) they are made equal to an arbitrary numerical value: $x + a - b = c$; $a - x - b = d$ (closure). ii) They are treated as equations $x = a - b$; $x = -b - a$. iii) Conjunction of dissimilar terms: $x + a - b = (a-b)x$; $a - x - b = (a + b)x$; $a - x - b = ax + b$.

4. Inconsistency in the use of algebraic language. When equations are solved and there is the possibility of a negative solution, the following is found: i) school methods for solving equations are not used. ii) The structure of the equation is altered in order to obtain a positive solution. For instance, the equation $x + a = b$, $a > b$ becomes $a - x = b$.

5. Preference for arithmetic methods in the resolution of word problems.

6. Ignorance of negative solution in problems. The students solve the word problem without expressing the solution in negative terms. They use verbal language in order to give a positive answer.

Profile D

1. Permanence in the additive domain and no-intervention of the multiplicative rule of signs in additions and subtractions with integers.

2. Recognition of the Triple Nature of Subtraction and the Triple Nature of the Minus Sign.

3. Correct operativity in the arithmetic and algebraic spheres.

4. Preference of algebraic language.

5. Predominance of algebraic methods.

6. Appearance of negative solutions in problems and in equations.

We conclude that the students of *Profile A* do not extend the numerical domain of natural to that of the whole numbers. The students of *Profile D* extend the numerical domain of natural numbers to that of whole numbers in the task presented to them.

As individuals acquire algebraic language, the extensión of the numerical domain becomes a crucial element for achieving algebraic competence in the resolution of problems and equations.

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[4] In the interview schedule, specific numbers always appear

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AN ANALYSIS OF SOME STUDENTS' CONCEPT IMAGES RELATED TO THE LINEAR FUNCTION WHEN USING A "GENERIC ORGANIZER"

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ABSTRACT

We are reporting here a part of a current research project which consisted of a study about the evolution of concept images related to slope and ordinate in the origin and formula-graph translations of the linear function using a "generic organizer". This study has been carried out with a group of 21 secondary education students of 15-16 years old, over 11 class sessions, combining the pre-test and post-test analysis with a case study of 4 students.

INTRODUCTION

There are arguments in the field of Mathematics itself, Cognitive Psychology and Mathematical Education which emphasize more and more the importance of images and visual reasoning in the teaching-learning of Mathematics. Learning environments using computers open up a very useful opportunity for this purpose. Since their visual power is unquestionable and furthermore it is possibly to work simultaneously with several representations; specially in the case of algebraic and graphic ones, as we can avoid algorithmic calculations. Our main goal is to arrive at a true vision of what kind of understandings students develop in this environment using software. In this work we have used the *A graphic approach to the calculus* software (Tall & al., 1990) for the study of linear function. The class teacher carried out the activity in the lesson in order to allow the researcher to externally observe the sessions.

The study aims we are reporting here (Garcia, 1995) can be summarised as follows:

1 - Analyzing whether ,this software is a suitable tool for the direct formula-graph translations of the linear function, by taking into account the parameters, without using the tables.

2 - Explicitly including those translations in the study of the straight lines in the classroom according to the advice of some researchers.

3 - Studying the evolution of the students' concept images in relation to the slope of the straight line and the origin ordinate.

4 - Checking previous studies' conclusions related to the linear function.

5 - Making a preliminary study for further and wider research in the future.

A THEORETICAL FRAMEWORK

The theoretical justifications rest on three fundamental precepts:

a) The "generic organizers" (Tall, 1993). These have two special features:

- the principle of selective construction: these programs lead to a new way of learning in which the individual can concentrate on the mental constructions of relations which are important for conceptualisation, while the computer takes care of the algorithmic routine. It is the teacher who selects particular mathematical concepts or processes while the program does the rest.

- the interface between the human operator and the computer. These programs considerably reduce the physical process for the user and extends those of the machine.

b) Models of cognitive processes implicit in the learning of some mathematical concepts.

We have principally followed the ideas of Tall and Janvier. Tall & Vinner (1981) and Vinner & Dreyfus (1989), distinguish between the "concept definition" and "concept images", the former is the formal definition (the personal definition or the mathematically accepted definition, which do not have to coincide), and the concept image is the set of the mathematical objects which the student considers examples of the concept and which is not necessarily the same as the set of the mathematical object determinated by the definition (formal).

Janvier (1987) talks about "representations" which he classifies as "external" and "internal"; the latter corresponds to concept images. Tall (1991) has coined the term "procepts": these are the combination of processes and concepts, in which an object can be used in a dual fashion as a process and as a concept, as in the case of a function. In a similar way, Douady (1986) talks about the "tool-object dialectic".

Furthermore, Janvier defines the "modes of representations"; in the case of functions we have the modes of graph, formula, table and verbal description. The translations processes are psychological processes which we use for changing from one mode of representation to another one, for example from an equation to a graph. There are direct translations which are those which are carried out without halfway stages, such as formula to graph, and there are indirect translations. Douady talks about "settings" and "changes of settings", although there are some differences.

In teaching strategies we find a certain degree of consensus: several authors advice that the principle behind teaching should be to approach the translations in pairs; they also agree that we should give students rich experiences which allowed them to extend their concept images; a graphic software, and in particular a generic organizer, is very useful for this type of translations, since one can simultaneously visualize at least two of them.

c) Studies on concept images associated with linear function

Research both with a computer and without it coincide in several aspects. Azcarate (1990) points out three profiles, "geometric", "operational" and "functional" in the concept image of the slope: the first one is associated with the words inclination, angle, the second one, which is more academic, is associated with the coefficient of "x" in the explicit expression of the function, and the third one is associated with the incremental quotient of variables. As far as the difficulties in the procedures of obtaining the slope are concerned, the results coincide with the studies carried out by computer, which are described below.

The results of Moschkovich (1990) and Schoenfeld & al. (1990), although they were carried out in different contexts, coincided in various aspects, in particular:

- the origin of the slope, its magnitude and the "y-intercept" are not independent, the variation of one means the variation of the other.

- disassociation of the calculation of the slope, the concept definition and its graphic meaning.

In Schoenfeld's study the different levels of knowledge structure in graphs and straight line equation are analyzed, finding that there are significant differences between those done by an expert and those done by a learner. They described three levels: "macro-organization", "concepts" and "fine-grained knowledge structure", the latter being formed of "atoms" of knowledge, this is where the connections between them are established. Specially in the case of straight lines, they denominate them "cartesian connection" or that which characterize the link between the algebraic world and the geometric world. In case of study they observed that as well as these three levels, there was a fourth, tied to specific contexts.

METHODOLOGY

Our experiment has been carried out with a quasi-experimental methodology combined with case studies, as follows:

Subjects: 21 secondary school pupils aged 15-16 who had never worked with a computer in the mathematics classroom.

Classroom management: 11 class periods in the computer room (not their usual classroom) with 10 PCs and the *A graphic approach to the Calculus* software (Tall & al., 1990). Handouts with guided activities were given to the students to use when working with this software.

Research instruments:

a) a preliminary pre-test taken by two other class-groups of the same level,

b) pre-test and post-test on different items: preliminaries, slope concept definition, questions about the formula-graph translation processes,

c) semi-structured interviews of four students, before and after the instruction in order to complete the information obtained in their tests and to analyze the degree of integration between the algebraic and geometric objects. Both the post-test and the final interview included questions to transfer these concepts into new situations,

d) external observation in the classroom: teacher's and investigator's classroom diaries and tape recording of the sessions,

e) a questionnaire presented to 13 mathematics teachers asking them to prepare some handouts for lessons in the same conditions as in the work object of this study and answer questions about their methodology when working with computers in the classroom.

DATA ANALYSIS

a) Pre-test and post-test analysis

As a main strategy used for this analysis the systemic network-charts were used. These were based on the "systemic networks" proposed by Bliss & al. (1983).

We started classifying the test items in two large groups: those related to the slope and those related to the "y"- intercept. After that, we summarized the students' answers to each one of the items into short phrases and with these we built the systemic network-charts (see an example, figure 1). Subsequently we made a double analysis, individually and in groups, of the data, prior to and after the instruction. Preparing evolution charts (see figure 2) enabled us to compare data and show the students'conceptual changes and the influence of the instruction. Finally, we made the summary charts, establishing the students' profiles and the formula-graph translations processes, together with the profiles' evolution.

During all the stages of the analysis, we used the class diary in a descriptive way in order to contrast the data.

b) Analysis of the case study

The methodology consisted in analyzing the students' answers before and after the instruction and their evolution. With each one of the four students we proceeded as follows:

1.- We made a statistical study of their answers in the pre-test and in the post-test examinations, in order to explore their "graphic", "algebraic" and "cartesian connection" profiles.

2.- We made an analysis of the initial and final interviews in order to know the concept images related to the geometrical and algebraic objects, and their degree of integration.

3.- We made connection charts between their algebraic and geometrical understandings to have a general overview of their concept images prior to and after the instruction.

4.- We made charts for each one of the students to see their evolution (their "cartesian connection" prior to and after the instruction) in a global way.

ANSWERS				STUDENTS	%
inclination (angle)	with respect to the axis/axes	of abscisas	----	LR, GD, CT, IM, EF, MP	57.1
		of coordinates	-----	RB, IP, JL	
			when the inclination is positive it is upward, if negative it is downward	SR	
	of the straight line	the angle formed with the ordinate axis & abscisas		YZ	
		may be upward or downward		YR	
operatio-nal	the value of the number that multipies to "x" /variation rate			LT, RQ	9.5
line/ straight	of a graph which is drawn	-------		MM, MN	19
		is found by calculating the rate of variation		RQ*	
	which pases through the point of origin and always has to be straight			MG	
"pendiente/ depen-diente" (*)	the "y" variable depends on the "x" variable			FT	4.8
no response				ES, GT, EM	14.3

Figure 1: Item 1 Post-test: the slope concept definition

(*) "pendiente" (slope) and "dependiente" (dependent) are easily confused in Spanish

	PRE-TEST	POST-TEST
inclination (angle)	ES, **JL, LR, MP, YZ, YR, GD**	**LR, GD, CT, IM, EF, MP, RB, IP, SR, JL, YZ, YR**
operational	RQ	**LT**, RQ
line/straight line	**LT, CT, IP**, MN, MM, FT, MG	MM, MN, MG, RQ*
confused "pendiente" with "dependencia"	**SR, RB**	FT
graphical description	GT, **EF**	
no response	**IM**, EM	ES, GT, EM

Figure 2: Answer evolution of the slope concept definition

CONCLUSIONS

Specific conclusions

1.- There are three students' profiles related to the graph-formula translation processes, namely: a) "algebraic"; most students belong to this profile: those who can translate directly from a formula into a graph, but have problems translating from a graph to a formula; b) "graphic", those students who can translate directly from graph to formula, but have difficulties translating the formula into graph; c) "cartesian connection", those students who can make the double direct translation graph to formula and formula to graph; this profile is the rarest.

2.- The students' profiles studied by Azcárate (1990) are confirmed in the concept definition of slope. A new profile appears: students who cannot differentiate between the slope and an inclined line.

3.- It is quite difficult to unify the concept images of the slope, the rate of variation (to compute the algebraic formula $y=(y_2-y_1)/(x_2-x_1)$), and the coefficient of "x" of the equation $y=mx+b$.

4.- In the case study, two students acquired the "functional" concept image of slope, that is to say, as an incremental quotient of variables.

5.- In some cases an overlapping of learning's relating to the slope occurred. The new concept images overlapped the old ones.

6.- The ordinate in the origin and the independent term of the equation are better integrated than the slope with the coefficient of "x".

7.- The psychological processes involved in the translation of graph into formula are more complex than those in translating formula to graph, as many studies have described.

8.- Schoenfeld's & al. (1990) and Moschkovich's (1990) studies are confirmed: a) the dependence of the parameters "m" and "b" in the equation, b) the variation rate is the absolute value of the quotient of the variables, c) in many cases the students' concept image of slope is not associated with their concept definition of the slope, d) one student in the case study considered that the straight line's increase or decrease was related to the place where the line came from.

9.- Some of the students do not see the co-ordinate axes as straight lines: they are only places where points are represented.

10.- Some of the responses are influenced by the teacher's discourse in the classroom. The students answer what has been said, not what they really know.

11.- Where a change of the scale was produced by the software, the students would think that there were in fact two different straight lines.

12.- Using this software has resulted in the positive evolution of the students in 71% of cases.

Methodology conclusions

1.- Interviews were conclusive in the profiles research. They revealed more and complete data than the written examinations.

2.- The role of the computer in the classroom could not be evaluated properly by way of pre-test and post-test examination, or by way of isolated interviews without the computer. Different instruments will have to be found for the evaluation of the computer role in the classroom.

3.- The systemic network-charts proved to be very adequate in the qualitative analysis of the data.

Teaching conclusions

1.- We usually let the students make the cartesian connections without working with them explicitly. Obvious things for teachers are not necessarily so for the learners who are presented with them for the first time.

2.- The understandings the students seem to have in an examination or during isolated interviews is not so deep as we think.

3.- It is necessary to be aware of the existence of variable and invariable features where scale changes take place, in order to get a complete understanding of the graphic representations.

SOME OPEN QUESTIONS

1 - Will the "algebraic", "graphic" and "cartesian connection" profiles remain when studying other functions?

2 - Could these profiles be considered as reminders of the existence of mixed ability groups in the classroom?

3 - Is the "cartesian connection" a necessary condition in order to use functions as objects which can be manipulated and to acquire the flexibility required to see them both as processes and as concepts as it has been described by Tall (1991)?

4 - Will the concept images related to other polynomial functions behave like slope, the rate of variation and "m" in the linear function?

5.- Would the conclusions be the same if the graphic calculators were used?

6.- Is the existence of mixed ability groups an important factor in the lack of the integration of the computer in the classroom?

7.- Would the class management be improved with project work rather than by using isolated activities?

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LEARNING PROPORTION USING A COMPUTER ENVIRONMENT IN THE CLASSROOM

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Abstract

The present research analyzes the efficacy of a computer environment designed to help students solve proportion problems at the introductory level. The strategy addressed in the instructional material is the computation of the ratio as a tool to solve problems of proportion (missing value and comparison). Results show the efficacy of the environment in (1) fostering conceptual learning and developing problem solving skills, (2) maintaining this learning and skills over a year, and (3) facilitating transfer to other domains.

INTRODUCTION

The use of proportion is a powerful tool to solve a variety of physical problems and to understand situations that require comparison of magnitudes. Proportional reasoning is also an important mathematical acquisition related to the conceptual field of "multiplicative structures" and a central topic in the secondary school curricula. On the other hand, proportional reasoning is known to be not only difficult to learn, but also very complex to teach (Karplus, Pulos, & Stage, 1983; Noelting, 1980a, 1980b; Sokona, 1989; Vergnaud, 1983). The difficulties students show when they solve proportional reasoning problems have been structured according to the types of variables involved (Harel, Behr, Post, & Lesh, 1993) into task-centered variables (problem structure, numeric complexity, an integer or a noninteger ratio, etc.) and context variables (familiarity with the physics principles underlying the problems, kinds of multiplicative relationships, kinds of units, etc.). The rate of subjects' problem solving success has been found to be closely related to the type of variables involved in the problems.

From the instructional point of view, the above mentioned variables should be taken into account. A sequence that began with the simple problems (direct, integer ratios, values of first measure larger than second one) and progressively presented the complex ones (inverse problems, non integer ratio, comparison problems, etc.) should be designed.

Several studies have pointed out the massive presence of *additive procedures* when the students solve proportional reasoning problems (Gómez-Granell, 1989; Grugetti, 1993; Karplus, et al., 1983; Noelting, 1980a, 1980b; Sokona, 1989; Vergnaud, 1983). Additive solutions seem more intuitive than multiplicative and less dependent on direct instruction (Resnick, 1986). It is fundamental that instructional designers address the students' explicit and conscious effort to deal with relationships linking different measures. This would help them understand the meaning of ratio that appear in proportional problems and avoid additive procedures.

The literature in the field also shows some multiplicative procedures closely related to instructional strategies. For instance, *cross-multiplication* (used to test for equality of ratios), *converting into equivalent fractions with a common denominator* (used to compare unequal ratios) or *rule-of-three algorithm* (to solve missing-value problems). These procedures are meaningfully used by only a very small subset of early adolescents students (Karplus, et al.,1983). For these students the problem is to understand the meaning of the calculations they mechanically make in order to know in which cases the algorithm can be applied.

The *calculation of the ratio* (calculation of the two ratios to contrast them in comparison problems, or calculation of a ratio as a well defined step to find the unknown in missing-value problems) is the more common multiplicative procedure. There are two variants within this procedure: The first one (scalar solution) calculates the ratio comparing the two quantities of the same measure space (for instance weight1/weight2, or price1/price2). The second one (function solution) calculates the ratio comparing one quantity with its correspondent quantity in the other measure space (for example weight1/price1 or weight2/price2). The literature shows no preference for one the other among students (Karplus, et al., 1983, Vergnaud, 1983). It is very probable, that the ultimate choice between these two procedures depends on the values of the problem and on the kind of relationships between the two measure spaces. Students tend to calculate the easier ratio and choose the functional solution when the functional relationship is facilitated by the contextual meaning.

From an instructional point of view, the functional procedure has the advantage of dealing with a ratio that allows a more solid interpretation. In some cases, the functional ratio can define a new measure (for example speed) with its own meaning. In other cases (price and weight, price and length, volume of fuel and distance, relative proportion of two components, etc.) the functional ratio leads to the "unit ratio" concept that allows the understanding of the quantity of one measure that corresponds to the unit of the other measure (for example kg per dollar, or liters per km, or kg of sugar for each kg of strawberries, etc.). The meaning of this ratio can help students understand the procedures applied to solve proportional problems. On the contrary, scalar ratios only indicate the relationship between two quantities of the same measure space but they can be hardly interpreted as a meaningful relation. Therefore, it is essential to elicit the students' awareness of the advantages to calculate the functional ratio to solve proportional problems and to make the unit ratio meaning explicit to them.

The above results have been taken into account to design a computer didactic sequence to work proportion at the introductory level. The general goal of the present study is to evaluate the efficacy of this computer program in terms of the progress in solving problems of proportion. The specific goals of the study are: a) to assess progress in understanding the concept of proportion through problem solving; b) to assess permanence in time of this progress, c) to assess transfer to other content domains, and finally d) to analyze subjects' strategy change after instruction.

METHOD

Subjects

Two intact classes of 21 students participated in the study. They were first year secondary school students of a middle class urban environment. One of the classes participated as the control group and the other as the experimental group ($\underline{M}_{control}$=13.1, range 12;3-14;9 and $\underline{M}_{experimental}$=13.7, range 12;3-14;1).

Procedure

The design consisted of 11 sessions, four corresponded to tests and the other seven involved a didactic sequence (see Tables 1 and 2). The pretest and the first posttest were done a day before and a day after the didactic sequence, respectively. The second posttest (*transtest*) was designed to assess transfer of proportion problem solving skills to different domains (12 months later). The third posttest (*permest)* was designed to assess permanence of skills over time (15 months later).

Table 1. Time Sequence of Pretest, Posttest, Transtest and Permtest.

Timing:	February 94		March 94	May 95	June 95
Control group:	pretest	Noncomputerized Sequence	posttest	transtest	permtest
Experimental group:	pretest	Computerized Sequence	posttest	transtest	permtest

Table 2: Didactic Sequence

session 2: Computer environment[1] "Ratio and Proportion", 8 exercises
session 3: Classroom activities. Introduction of the concept of proportion
session 4: Classroom activities. Strategy of finding the unit value
session 5: Classroom activities. Objects indirect measurement
session 6: Playground activities. Objects indirect measurement
session 7: Computer environment[1] "Ratio and Proportion", 9 exercises
session 8: Classroom activities. Content review.

Material

The computer environment consisted of 17 exercises to be completed in two sessions. The level of the exercises was introductory, the semantic content was the height of different objects and their shadows, and they were sequenced from easy to difficult according to the literature (Vergnaud, 1983). There were two main types of problems: missing value problems and comparison problems. The computer environment was designed to provide the student with appropriate feedback. The most important aspect of this feedback was to focus the student towards the computation of the ratio.

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[1] Only for those subjects in the experimental group. Subjects in the control group worked the concept of proportion using an equivalent studying material

The pretest, posttest, and permtest were identical in terms of structure and similar in terms of numbers. They consisted of 12 exercises. Seven of them were missing value exercises and the other five were comparison exercises. In contrast with that, the transtest consisted of 20 exercises, ten embedded in a physical domain (density: g/cc, and speed: km/s) and ten embedded in a numerical domain, (sell and buy: amount/$).

RESULTS

The results section is divided into two blocks. The first one-General learning outcomes-refers to goals (a), (b), and (c), specified in the introduction and the second one-Qualitative analysis of problem solving strategies-refers to goal (d).

General Learning Outcomes

To assess progress in problem solving right after the treatment, a year later, and to assess transfer of such progress to other semantic domains a comparison of mean scores of all subjects in each test was performed. The distribution of means and standard deviations is represented in Table 3 and the pattern of change of such means in Figure 1 for the four tests together.

Table 3. Means and Standard Deviations of Pretest, Posttest, Transtest, and Permtest in the Control and the Experimental Groups.

Test[a] Group		Pretest	Posttest	Transtest	Permtest
Control					
	M	4.2	5.6	7.0	6.7
	SD	2.9	4.1	4.0	4.6
Experimental					
	M	4.4	8.9	9.1	9.9
	SD	3.7	3.5	3.1	2.4

Note[a]. Means refer to a total of 12 possible correct answers. Results in the transtest were corrected from a total of 20 exercises.

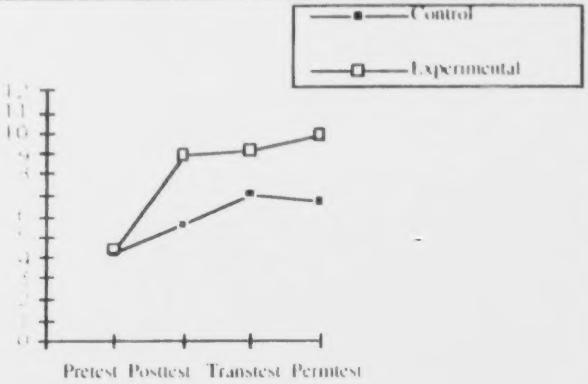

Figure 1. Mean Number of Correct Answers in the Pretest, Posttest, Transtest, and Permtest

We initially present results comparing subjects' performance in each test irrespective of the type of problem solved correctly, comparing mean scores in pretest, posttest, and permtest. The first analysis that was performed was a comparison of means between groups for each test (split plot factorial). Interaction between time and group was significant, F(3, 120)=65.5, p<.05. Figure 1 shows that there was an improvement in both groups, but the rate of this improvement was much higher for the experimental group. Therefore, ten post hoc contrasts comparing means in each test for each group were performed. From this results we can conclude that: (a) there were no differences between groups a priori (Contrast 1:F(1,120)=3.5). The two groups were equivalent before the treatment. (b) The effect of the treatment on the two groups was significantly different (Contrast 2: F(1,120)= 10.73, p<.005[2]). c) Differences between the control and the experimental group mentioned in contrast 2 remained a year later (Contrast 3:F(1,120) =8.6, p<.005). (d) Differences between control subjects' performance in the pretest and posttest were nonsignificant (Contrast 4:F(1,120)=3.5). The treatment in the control group caused only a slight improvement that did not reach significance. (e) Differences in subjects' performance in the pretest and the permtest were significant (Contrast 5:F(1,120)= 10.73, p<.005). Subjects in the control group not only maintained, but also slightly increased learning gains over time (Contrast 6:F(1,120)=1.92). f) Differences between experimental subjects' performance in the pretest and posttest were significant (Contrast 7: F(1,120)=34.5, p<.0001). There was a clear effect of the treatment in the experimental group, that was maintained over a year (Contrast 8:F(1,120)= 53.41,p<.0001). (g) There were no significant differences between the posttest and the permtest for the experimental group (Contrast 9:F(1,120)=2.05). This could be explained by means of the high rate of correct answers reached by the subjects in this group (8.9 in the posttest and 9.9 in the permtest; see Table 3). (h) Finally, the last contrast, although marginal, shows a constant difference between the control and the experimental groups that did not exist a priori, originated in the treatment and maintained over the three posttests. This difference is minimal in the transtest (Contrast 10: F(1,120)=4.34, p=.038[3]).

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[2] Significance is calculated applying Bonferroni for ten contrasts: for α 0.05/10

The next step in the analysis is to look for the cause of the differences found in this section. In order to do that, a new split plot factorial was performed in which the score in the tests was divided into two parts according to the type of problem (missing value and comparison). In this new comparison of means, subjects' mean number of correct answers in the missing value problems (which had to be transformed into an "out of five" number) were compared with subjects' mean number of correct answers in the comparison problems , in both the control and the experimental groups.

The statistical results yielded an almost significant three-way interaction, between group, time and type of problem, F(1,80)=3.25, p=.07. Figure 2 shows the pattern of change of the means for this analysis and illustrates the interaction between the three factors. The tendency observed of the missing value problems getting a higher rate of correct answers, especially in the experimental group, became inverted in the posttest of the comparison problems. The rate of correct answers in comparison problems for the experimental group was the highest.

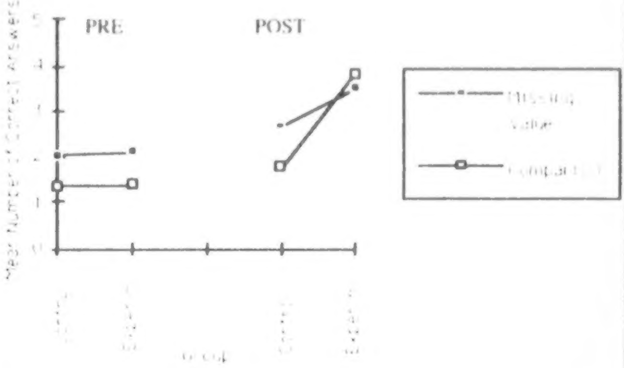

Figure 2 Mean Number of Correct Answers in the Pretest and the Posttest Comparing Subjects in the Control and Experimental Groups with respect to Missing Value and Comparison Problems

That is, the computer environment seemed to help the students in the experimental group with the problems of comparison (8 through 12), rather than with the missing value problems (1 through 7), where an unknown value had to be calculated. We argue that the reason for that could be that the main goal of the exercises in the environment was the awareness of the need to calculate the ratio to solve problems of proportion. This will be commented further in the Qualitative Analysis Section where the different strategies used

[3] Marginal value due to application of Bonferroni

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by each subject in each group, and how these changed over time, from the pretest to the posttest and from the posttest to the transtest are described.

Qualitative Analysis of Problem Solving Strategies

Subjects' problem solving strategies were classified into four categories: *additive, ratio, rule of three* and *others*. The last category included those strategies that were either used very seldom (i.e. cross multiplication) or that could not be well defined. There were two percentages whose values showed a clear pattern of change. The first one involved the category of the additive strategy, which got reduced from the pretest to the other tests, with this reduction being striking in the experimental group (from almost a 30% in the pretest to a 5% or less in the other tests). The other category that deserves attention is the one involving the ratio strategy. It showed only a slight increase in the control group over the sequence of tests, but a spectacular increase (from 41% in the pretest to 88.1%, 85.5% and 90.0% in the posttest, transtest-physics and transtest-numeric, respectively) in the experimental group. Another characteristic of the control group performance was the application of the rule of three strategy, which was present in all tests although low. Subjects in the experimental group in contrast, never applied it except for one subject in one problem.

In addition to these categories, and in order to interpret the transfer results, another subcategory into the ratio strategy was added to subjects' performance in the transtest. It refers to the use of units as tools to calculate the ratio. In the control group there was a 2% of strategies in both, the physics and the numeric problems. In contrast, the experimental group applied it 24% of the times in the physics.

Therefore, the results show that although subjects in the control group improved gradually, never reached the level of progress achieved by the subjects in the experimental group. This progress was striking in the posttest, the difference was maintained a year later and the gains were transferred identically to both a physical domain and a numerical domain When the learning outcomes were analyzed according to type of problem, the analysis showed a clear improvement in the comparison exercises. A qualitative description of the strategy change showed that the this improvement was due to a reduction in the additive strategy, and an increase in the strategy of calculating the ratio. In addition to that, another reason for the experimental group performing better than the control group in their transfer ability was the combined strategy using the unit values to calculate the ratio, strategy that implies a good understanding of the concept of proportion.

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DEVELOPMENT OF CONCEPTS OF THE ARITHMETIC AVERAGE FROM HIGH SCHOOL TO UNIVERSITY

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The objective of this study was to explore students' strategies for solving average problems and to see how these strategies change over high school, college and university. Students in high school (with or without having received a lesson on the average), in college and in university were tested. The results show that achievement does not necessarily follow the years of study particularly in problems where conceptual understanding is needed. The strategies used are various and are distributed differently amongst the groups. College and university students tend to use algebraic strategies and are better at finding an average of grouped data. Strategies preferred by high school students use the total of the data and bring more success in "reverse" problems.

Average is a widely used concept encountered daily, particularly in the life of scholars. However, despite this fact, it seems to be a more difficult concept to understand than it might have appeared at first. Previous studies have shown that a superficial understanding of the mean is widespread amongst children and adults. Most people can compute an arithmetic average in simple situations but are not able to use it otherwise. Apparently students see the arithmetic mean as a computational result devoid of any meaning (Pollatsek, 1981; Carpenter et al., 1981; Garfield, 1988a, Lappan, 1988). Recently, Cai points out that even if most sixth-graders were able to calculate an average, only half show some conceptual understanding of the concept (Cai, 1995). Students and even adults are prone to mix numbers and place quantities in a formula without understanding the concept.

Some of the difficulties encountered in average problems are the ones found generally in problem-solving situations, i.e., decimals (Travers, 1986), terms used (Carpenter et al. 1981). Other are specific to the concept: Strauss & Bichler (1988) identified seven properties of the mean. The results showed that two of the properties identified still pose some difficulties for 12-14 year-olds: taking the zero into account and the representativeness of the average. Russell and Mokros (1990, 1995) explored children's conceptions about arithmetic average. The results showed that children do have a reasonable idea of representativeness and see the mean as an indicator of centre although there is some confusion with the mode, midpoint or reasonable value. The introduction of an algorithm as a procedure seems to short-circuit the reasoning of the children and cause them to give up their sound intuitions. What happens and what changes occur in solving average problems? Does the situation correct itself later on?

The objectives of this study were

(a) first, to verify if students of different grades were able to solve average problems,

(b) secondly, to see if the strategies used change over high school, college and university.

The study

A total of 241 students served as experimental subjects. To observe possible development, we chose four groups of students: two groups of third year high school students (age 14), of which 40 students were tested before the lesson on average (HS^-) and 41 after the lesson on average (HS^+), a third group (C) of 74 first year college students (age 17^+) currently enrolled in an introductory statistics course and a fourth group (U) of pre-service high school mathematics teachers with a college background in science (age 20^+).

In order to explore the research questions, we designed a test of six tasks. It was administered in four different versions, in which the order of the tasks was permuted to minimise order effects. For every task, explanations were required and space was provided on the sheet. The students were given all the time necessary to answer; they took at most 35 minutes.

Description of the tasks

In this paper, we will discuss four of the tasks. Numbers were kept as simple as possible to lessen computational difficulties and contexts chosen were familiar ones.

Candies[1]

> Lucy, Jack and Paul get together for a party. Each one of them brought a certain number of candies. In fact, they brought an average of 11 candies per child.
>
> How many candies has each child brought?
>
> Lucy __________ Jack __________ Paul __________
>
> Is this the only possibility? Explain how you get your result.

The *Candies* task was meant to see if the students had a basic idea of what an average represents and if they could find a solution without directly applying the usual add-and-divide algorithm. The problem is very simple and requires listing three possible data once the average is known. We expected to see solutions where the mean would be the middle quantity or all the data would take the mean value. Given explanations would offer more information on strategies used by the students.

Zero

A fourth child comes to the party without any candies. What is the average of candies per child? Explain how you get your result.

This sub-question of *Candies* aims at verifying if subjects older than the one tested by Strauss & Bichler (1988) take into account a value of zero in calculating an average.

Cookies

The number of broken cookies in a bag was observed and the following data were collected:

Number of broken cookies	Frequency
2	5
4	8
6	7
9	2
13	1

In average, how many broken cookies per bag are there?
Explain how you get your result.

The data were presented in the form of frequency tables quite similar to the ones seen in textbooks. The numbers were chosen carefully to allow different misconceptions to appear. The correct mean, 5, is different from both middle terms (number of biscuits, 6 or central frequency, 7) and the modal value, 4 or its frequency, 8.

Other errors previously observed were expected: finding the mean of a single column, dividing the total of cookies by 5, etc. Answers to this question would also reveal the acquisition of a computation algorithm.

Homework

In Mrs Girard's class, students who have an average of 6 out of 10 for their four homeworks of the week get a "vacation" for the week-end. After 3 homeworks, Mary has an average of 5. What mark does she need to get in order to have no homework for the week-end? Explain how you get your result.

This is a "reverse" question dealing with a weighted average. It asks for a better understanding of the average concept. The context is very familiar: average is easily associated with marks in school. However marks are not concrete objects like cookies or candies.

The results

Before looking at the strategies, we will describe the results in term of performance. The *Homework* problem was expected to be the most difficult and did present difficulties for 52% of the subjects. However, the *Cookies* problem had the lowest score (44%), especially in the two high school groups (20%, 46%).

The problem with the highest score was *Candies*. We wanted to verify that the students had an idea of what an average means; 79% knew, some did not answer (16%) or gave a wrong answer(5%), most of them being in the University group. Surprisingly, performance decreased from HS$^-$ (88%) to University (70%). This might be an indication of a certain helplessness in front of a situation where direct computing is inefficient.

About 50% of all the subjects were able to answer the *Zero* problem but this result increased to 70% if we take into account 20% of the students who gave an integer answer rounding off 8,25 to 8. Some subjects had difficulty seeing the average as a fraction with no counterpart in reality. Results for both HS groups (78%) were higher than the ones found for the C and U groups (66%).

Performance[2] for the *Cookies* task increased with instruction: 22.5% for HS$^-$, 49% for HS+, 63% for C and U. There was an improvement with age and instruction in the computing of the average of grouped data (Cookies). However when the task asked for a better understanding of the average concept, like in the *Homework* problem, scores did not increase in the same manner. The best performance was found in the HS$^+$ group (63%) more than the U group (56%), the college group (42%) and the HS$^-$ group (40%). This is surprising and needs some investigation. If no distinction between the strategies used was made, 12,5% of the subjects answered 7 which is the unweighted average.

In summary, these results corroborate the fact that although students do have a good idea of what an average is in simple contexts and simple situations (few integers, unweighted data), they have more difficulty when it comes to weighted average: The subjects answer correctly the *Candies* and *Zero* problems more than *Cookies* and *Homework* problems that ask for a weighted average. However, the performance for this task does not vary in the same way over the age groups.

The strategies

After a first look at the performance, we tried to explain the results by looking at various strategies produced by the students, to establish possible links with performance and with levels.

In ***Candies***, where the subjects had to choose the data corresponding to a given average, the following strategies showed up.

Sum of the data: "It has to add up to 33", or "the sum has to be 33" or "3×11 is 33"	S
Number of data: "When you divide it by 3, it has to give 11" or "x/3 = 11"	N
General explanation (verbal or algebraic): "the total of the candies divided by the number of children has to equal to 11" or "$\frac{x+x+x}{3} = 11$"	G
An arithmetic verification: $\frac{15+10+8}{3} = 11$	A
Levelling around the average value	L

The first ones, S, N, G, A, are variations of the computation algorithm or the "fair share model" but the emphasis is placed differently. The levelling (L) strategy, the only one resulting in no wrong answer is closer to the "balance model".

The distribution of the strategies differs between groups. Strategies used by the youngest students are in some way closer to the situation and less abstract than the one used by older students who prefer an algebraic representation.

Candies%	S	N	G	A	L
HS-	**28.5**	9.5	2.3	**21.5**	7
HS+	19.5	10.8	4.3	10.8	**28.2**
C	22.2	**21**	1.2	7.4	7.4
U	21.8	10.5	**24**	5.2	1
All	22,6	13,6	10,2	9,3	8,6

The strategies expressed in the ***Zero*** problem were similar to some of the previous ones except for the L which was discarded. The only important difference between the groups is again that the university students give a more general explanation (G), whereas the other groups prefer the N and A strategies. However, the *Zero* question unexpectedly revealed that about 23.5% of the HS subjects, 10% of the C subjects and 22% of the U subjects did not accept the average as a fraction with no counterpart in reality. The comments were: "8,25 means an average of 8 candies for each child and there is one left over" or "Three of the children have 8 candies and the fourth one has 9." Five subjects (4 in C) did not take account of the zero and said that "the average just stays the same". A few subjects used a proportional reasoning, or took the middle value (11÷2)

In the ***Cookies*** problem where the data were grouped, the Algorithm strategy (F): $\left(\frac{\sum x_i \times f_i}{n}\right)$ was the most frequently used (50%) and explains almost all the right answers even for the high school subjects that had not received any instruction on grouped data. Three subjects listed the data first and went on with the usual "add-and-sum" strategy. All other strategies led to incorrect answers. They are very similar "variations" of the computation algorithm, they use addition and division. The principal ones are the following.

Quotient of the sums: $\sum x_i + \sum f_i$.	QS	9%
Division of the total of a column by 5: $\sum x_i+5$ or $\sum f_i.+5$	D5	7.5%
Division of $\sum x_i \times f_i$ by $\sum x_i$	BB	7%
Division of $\sum x_i \times f_i$ by 5	SB5	5%

Again the distribution of the strategies differs from one group to the other. The observation of the strategies suggests that there might be difficulties in the interpretation of the table, in "reading" the data, which caused a misuse of the computing algorithm. We had expected to see answers that would reveal conceptions previously observed (Russell & Mokros, 1990) where the largest values or the middle terms would be chosen but only one answered "6 the middle value" and one more gave the modal value divided in half. Another said: "13÷2≈6 so the average is 6", where 13 was the largest data.

Cookies %	F	QS	D5	BB	SB5
HS-	22.5	**20**	15	2.5	5
HS+	44	9.7	**14.6**	4.8	2.5
C	56.7	2.7	6.7	**15**	5.4
U	60.4	**8.1**	1	2	6
All	50.2	8.7	7.4	6.6	5

In ***Homework**, the* most frequently observed strategy is the Totals-Difference (23.8%): "–An average of 5 for the 3 first homeworks makes a total of 15. –To get an average of 6 for the 4 homeworks you need a total of 24, 24 – 15 = 9, so you need 9/10 for the fourth homework". This strategy, like the S strategy in the Candies puts the accent on the sum of the data. The second one is the algebraic strategy (G) (13%) followed by the arithmetic strategy (8%). This last strategy seems to be a verification of an estimated answer, it leads to a wrong answer half of the time. Sum of ratio (6.2%) and levelling (5.3%) strategies come next. The Sum of ratio considers a mark as a ratio: "5 points out of 10", i.e., $\frac{5}{10}+\frac{5}{10}+\frac{5}{10}+\frac{9}{10}=\frac{24}{40}$.

Homework%	DT	G	A	SR	L
HS-	12.5	7.5	**15**	5	7.5
HS+	**40.5**	7.1	4.7	0	**16.6**
C	17.3	10.6	**14.6**	5	2.6
U	26.7	**21**	0	10.5	1
All	23.8	13	7.8	6.2	5.3

The results show the predominance of the DT strategy, specially in the HS$^+$ group which also uses the levelling strategy more often. The U group uses the algebraic strategy (G) almost as often and never an arithmetic verification (A). These two groups achieved a better score. The subjects of the two others groups, HS$^-$ and C, who had less correct answers employ the arithmetic strategy more and the DT less than the HS$^+$ and U groups.

Let us mention that an answer of 7 is produced in 12.5% of the cases. The fact that the given average is for the first three homeworks is not considered, regardless of the strategy used. For example: "$\frac{5+x}{2} = 6 \rightarrow x = 7$ or $\frac{5}{10} + \frac{x}{10} = \frac{12}{20} \rightarrow x = 7$" even with the levelling strategy "5 is one less than 6, so 7 is one more".

Discussion

This study examined four groups of students from high school to university. Performance on a task (*Cookies*) that is usual in school increases with instruction. Conversely, when the task asks for "reversibility" in a very easy form (*Candies*) or with a weighted average (*Homework*), high school students perform better. Difficulties in taking account of a zero in computing the average and considering a fraction as the average still remain in university (*Zero*).

Solution strategies were also examined. The distribution of strategies differs from one group to the other. Examining the Candies and Homework tasks, we see that high school students tend to use less abstract strategies, Arithmetic and Levelling, than older students who prefer an algebraic strategy (G).

In *Cookies* the algorithm strategy accounts for almost every correct response. Some subjects of the high school groups had never received any specific instruction for the average of grouped data yet they used the same strategy correctly .

Although some older subjects do use algebra correctly, greater success in problems asking for a more conceptual understanding was observed in the high school groups and is mostly due to strategies, like Sum in *Candies* and Totals-Difference and Levelling in *Homework*, exploiting the fact that for a given average the sum of the data is fixed. The frequency of this strategy should be taken into account in the design of teaching experiments.

Trying to explain the achievement of the HS+ in the *Homework* problem, we looked at the data and found that one class had a much better score (75% vs. 40%[3]) and utilised the Levelling strategy much more (23% vs. 0%) than the rest of the HS+. This result is probably due to the fact that the teacher explored this strategy in class.

Although this analysis answers some questions, it raises others. To what extent does instruction influence the strategies used? Some of the younger students succeed in finding the average of grouped data without instruction and more of the college and university students could not find a missing data in a weighted average problem. Teachers should provide various situations were the average is present and take into account the students reasoning before imposing the traditional computing methods that seem to hinder a deeper understanding of the average concept. The fact that the sum of the data is linked to the average, that finding an average is a way of equalising the data and also that the sum of the deviations to the average is zero should be discussed and exploited by the teachers.

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[1] The original problems were in French.

[2] All comparisons were statisticaly tested with paired or unpaired t-test with ($\alpha<0,05$)

[3] This class was a group of higher ability students

Selling mathematics: The language of persuasion in an introductory calculus course

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Abstract: Undergraduate students at North American universities often take introductory calculus courses unwillingly, and have a reputation of being difficult to teach. The author, noticing an unusual style of discourse in such lectures, linguistically analysed audio tapes of three first-year introductory calculus lecturers at a Canadian university and found discourse features in all similar to the "hard sell" pitch of salespeople. The author asks university mathematics lecturers to consider whether they wish to frame their teaching in terms of a persuasive sales pitch, and questions whether this approach will reach or engage first-year calculus students.

1. Focus of paper

Introductory calculus classes in North American universities are often disliked by both faculty and students. These classes are usually prerequisite for other non-mathematics undergraduate programs (biology, physics, economics, etc.) as well as for degrees in mathematics. Instructors face large classes which include many students who are not interested in calculus, students who may nonetheless feel desperate to get good marks in the course so that they can maintain a high academic standing. Both students and instructors also know that such courses have traditionally been viewed as "weeder" courses, in which a certain percentage of students will fail. Such conditions are likely to create an atmosphere of tension for teaching and learning.

As a researcher in the language of mathematics education, I noticed an unusual discourse style in these lectures. I hypothesized that this discourse reflected tensions inherent in the pedagogic situation. Upon closer linguistic analysis I found discourse features of persuasive language, and particularly of a salesperson's "hard sell" sales pitch in the introductory calculus lectures I studied.

2. Theoretical framework

There is a developing tradition that uses linguistics to look at various aspects of mathematics education. For an overview of the field of *language and mathematics* within mathematics education, see David Pimm's (1994) encyclopedia entry from The Second International Encyclopedia of Education. A variety of different types of analyses of math education from a linguistic point of view are provided in Durkin & Shire (1991), Pimm (1987), Pimm (1995), Lowenthal & Vandamme (1986) and Austin & Howson (1979).

Since the 1970's, a number of general educational studies have used linguistics to look at features of teacher and student talk in classrooms and the effect on different kinds of talk on learning (for example Cazden et al, 1972; Heath, 1978; Sinclair & Brazil, 1982; Stubbs, 1983). These studies are primarily concerned with questions of social class and power relationships in schooling. Areas of linguistics such as discourse analysis, pragmatics and stylistics have recently been used to look at mathematics education discourse -- for example, in Guillerault & Laborde (1986) and Pimm (1988). Roth (in press) uses discourse analysis, to look at the sociology of math students' learning.

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Rowland (1992) and Pimm (1987; 1995) apply pragmatics (particularly questions of *deixis*, or "pointing with words") to mathematics education.

In this paper, I will use linguistic theory from discourse analysis to analyze the discourse style used in introductory calculus lectures. By reframing the language of these lectures in terms of linguistic genres, I want to enable instructors to look at their teaching practices from a different perspective. My aim is to find unexpected connections through the medium of language and to explore some of their implications for educational practice.

3.1 Linguistic methodology: Discourse analysis

Discourse analysis refers to the structural analysis of stretches of "text" (in its broadest meaning) at a level larger than the sentence or utterance. The texts in question may range from spoken discourse to written texts and oral genres. The analytic methods that fall under the general heading"discourse analysis" are heterogeneous. However, of six contemporary approaches to discourse analysis described by Schiffrin (1994), three were used in this study: **interactional sociolinguistics, variation analysis** and **pragmatics.**

Interactional sociolinguistics, coming from linguistics, anthropology and sociology and influenced by John Gumperz and Erving Goffman, looks at cross-cultural differences in the contextualization of language and the understanding of language in context that results, and at the influences of social frames (for example, "the lecture" or "the radio announcer's monologue") on the language produced. **Variation analysis,** developed by sociolinguist William Labov, looks for formal patterns in text types, and describes variations and constraints on those patterns linked to the text type. **Pragmatics** has been variously defined (Levinson, 1983) as "the study of language usage"(5) and "the study of the relations between language and context that are basic to an account of language understanding" (21). Linguistic pragmatics includes the study of deixis, implicature, presupposition, speech acts, and aspects of discourse structure. In this paper, I will be using an analysis of pronoun deixis.

Deixis refers to the process of "pointing with words". Deixis studies the contextual referents for words like demonstratives ("this", "that"), pronouns ("I", "you", "it"), and adverbs of time and place ("then", "here"). I will be considering the contextual referents for the first person plural pronoun ("we", "us","our") in calculus lectures.

3.2 Literary/cultural studies: Genre analysis

The term "speech genre" was coined by Mikhail Bakhtin to describe "relatively stable types of utterances"(Bakhtin, 1986, 60). Bakhtin argues that the nature of utterances is shaped by their quality of *addressivity* -- that is"the quality of turning to someone", since every utterance is addressed by the speaker or writer to a perceived or imagined other and to that other's response. Bakhtin defines genres as the categories of utterance types, and stresses that genre can be analysed only by considering the whole of an utterance, including its thematic content, linguistic style and its addressivity.

In my analysis, I consider a number of features that characterize introductory calculus lectures as a genre. I hope to make useful analogies to other genres that may shed light on our use of these lectures in mathematics education.

4. Sample data and results

In this pilot study, I took an opportunistic sample of three lecturers teaching introductory calculus at Simon Fraser University in British Columbia, Canada. The three lecturers (pseudonyms Green, White and Brown) varied in age, gender, national and linguistic background. With the lecturers' permission, I obtained audio tapes of similar lessons in their courses, and undertook a linguistic analysis of their discourse features.

Analysis yielded the following list of features which I hypothesize to be distinctive to "introductory calculus lectures" as a genre:
1) A persuasive mode of speech with some parallels to other forms of persuasion (salesmanship, evangelism, political speechmaking) marked by the structure of the lecture, questioning style, and use of personal pronouns
2) A mode of speech that assumes a conversational partner who cannot respond (parallel to baby talk, radio talk, talk to invalids or pets), similarly marked by questioning style and use of personal pronouns
3) The use of tag questions to elicit agreement from the audience
4) The extensive use of metaphor and new lexical coinages, invented for the purpose of the lecture

4.1 The "hard sell" and the language of persuasion: fake dialogue

In a "hard sell", the salesperson's job is to forge an inexorable chain of logic that leads to one conclusion: that the prospect would be crazy not to buy what the seller has to offer. Typical tools used in the hard sell include demonstrations, making claims for the marvellous qualities of the product for sale, voicing possible objections or questions the prospect might have and then answering them with prepared responses, identifying oneself with the prospect's disbelief, and above all, creating an unstoppable stream of talk so that the prospect has no chance to consider any further objections or questions. Many of these features are also typical of the introductory calculus lectures I studied.

These features are also typical of other genres that involve "selling" or convincing - religious proselytizing, political speechmaking, "hard sell" advertising in all media. What is being "sold" is not always a tangible commodity; it may be a belief, an opinion, an idea. It is in this light that we can see the similarity between a math lecture and these other forms of persuasion.

Goffman (1981, 165) defines lectures in general as "an institutionalized extended holding of the floor in which one speaker imparts his views on a subject".The lecture genre, whatever its subject content, is already a mode of persuasive talk that tries to "sell" its audience on both the truth of the ideas presented and, implicitly, the authority and status of both the lecturer and the sponsoring institution as purveyors of truth and knowledge. Within the "persuasive" framework of the lecture, math lecturers use other discourse techniques that further the impression of lecturer as pitchman. All the lecturers in my study used the rhetorical question and the old sales trick of raising "fake" questions and objections on behalf of the listener and then answering them with prepared responses, as if the audience's real questions had then been addressed. Some examples from the data:

> Somebody's going to say, "Why not have infinity on both the input and the output sides ?" Well, why not ? (White)
>
> Now you want to know ,"Where is this function maximum?" And the first thing you might think to do, well, I know that the turning points or the points when that slope is zero is when something happens on the function. OK? (Green)

Schmidt and Kess (1986), in their study of linguistic persuasion techniques in television advertising and televangelism, cite experimental studies on the effect of rhetorical questions on persuasion. It seem that the effect varies with the listener's degree of involvement:

> In the case of rhetorical questions. . . high-involvement subjects actually found the use of this type of question distracting, making them less sensitive to the quality of argumentation than they were when the same arguments were presented as assertions. Low-involvement subjects, however, showed greater sensitivity to the quality of argumentation when the message contained rhetorical questions than when it did not. (Schmidt & Kess 1986, 31)

If calculus instructors see many of their students as unwilling listeners, "low-involvement subjects", it may seem natural to use "hard sell" techniques to try to involve them in the class.

The particular style of rhetorical questioning noted in the examples above, which we could call "fake dialogue", has been noted in other discourse styles as well. In a discussion of a political speech,Leith and Myerson (1989) note that "the speaker holds a dialogue not only with the audience, not only with opposite but absent voices, but also with the previous speaker at the conference" (Leith & Myerson 1989, 23). Goffman (1981, 241), in his discussion of radio talk, finds radio announcers engaging in "fake dialogue" with an imagined audience. A further example of "fake dialogue" has been noted by Ervin-Tripp and Strage (1985) in interactions between parents and pre-verbal children in which parents "interpret burps, or single- or two-word constructions as if children have elaborate intentions, confirming, expanding and elaborating them" (Ervin-Tripp & Strage 1985, 73). All these examples of "fake dialogue" have in common the fact that they are addressed by a speaker to an audience that cannot respond, either because they are not present , because they are incapable of responding, or because they are socially constrained from responding. The pedagogical question then becomes, should students learning mathematics be treated as if they cannot or must not respond ?

4.2 Tag questions

The lecturers made extensive use of tag questions ("Right?", "OK?" etc.) to elicit audience consent. These were used to elicit both agreement with a statement made by the lecturer and permission to move on to the next section of the lecture. Some examples:

> When theta is equal to zero I get this point. When theta is equal to pi over two you get this point. The in-between points, just take my word, OK ? (Brown)

> Now the denominator, this first term is approximately minus x. Not plus x! Minus x. It's a square root. It's got to be positive. X is negative. It's minus x, OK ? More or less. (White)

This use of tag questions is typical of the persuasive talk of both the salesperson and the conjurer. When an audience is addressed with such tag questions, and if there is no audience protest, it is assumed that there is tacit agreement with the point the speaker is making and that permission has been granted for the speaker to move on to another topic or example. Presumably the speaker is "reading" the faces of the crowd and seeing approval there. The audience is carried along on the lecturer's unstoppable stream of talk, since it is presumed that any rational person, having signalled acceptance of one link in the lecturer's chain of logic, will be led on inexorably to the next link, and the next, and finally reach the sole inevitable conclusion.

Further comparison can be made with the persuasive discourse of the conjurer. Several of the lecturers in the study used a magician's rhetorical technique: they slowly and labouriously built up audience approval of obvious statements (like the conjurer's "test" of his materials with a volunteer from the audience:" Is this a solid table? Pass your hand underneath it. No holes in this scarf?" etc.), then, like a conjurer, performed a series of rapid and complicated moves until something unexpected appeared. The lecturers hinted at the mechanics of illusion -- disappearances, levitation, things mysteriously expanding or shrinking -- which are metaphors common to the magician and the mathematician. Like conjurers, they made marvellous claims or set challenges to the audience in the course of their lectures, then carried out demonstrations to prove their claims were true:

> So what I'm saying when I say " the limit of one over x squared is plus infinity" is, really, I'm saying this: You give me a high horizontal line, and I'll show you how to get the graph to stay above that line. (White)

> You give me any number no matter how big a number you give me, I can show you how to make one over x squared even bigger than that, and stay bigger than that. (White)

4.3 Referents for the pronoun "we"

Pimm (1987) has discussed the use of the first person plural pronoun (we, us, our) in the mathematics classroom. I will apply some elements of his analysis to the non-standard uses of *we* which appeared frequently in these math lectures, and draw some conclusions about the use of *we* in persuasive discourse.

Pimm outlines four familiar contexts in which the non-inclusive *we* (a *we* that does not really include both the speaker and the listener) is used. These are baby talk ("We're getting you dressed, aren't we"), hospitals and doctor's surgeries ("Now we are going to take our temperature"), formal discourse ("In our attempt to analyze addition we are thus led to the idea ...") and school use ("Susan, we never bite our friends") (Pimm

1987, 67). Ervin-Tripp and Strage (1985) note that this feature of the baby talk register is also found in speech to pets, dolls, hospital patients, lovers, and the elderly and hypothesize that "they may indicate affection or protectiveness of the weak" (Ervin-Tripp & Strage 1985, 72). Pimm concludes that this atypical use of *we* may indicate a hearer who is unable to respond (a use that is functionally related to the "fake dialogue" discussed earlier), or may signal a relationship of power and dependence and carry an implicit message of condescension, indicate social convention being conveyed, and/or point at the existence of an "in-group" of mathematicians who carry authority within the field of mathematics. Some examples from this study:

> OK, now we're going to get into lots of trouble I think if we start doing this because all these things -- we're not going to be able to control the variables very well. (Green)
>
> Now let's see if we can see what's going on and if we can't , we're going to have to do more to it. (White)
>
> But we will try to show you people a better way of describing a curve. (Brown)

There is a good deal of confusion throughout the lectures about exactly who is included in *we*. On various occasions it refers only to the lecturer, or to the lecturer and the students, the lecturer and the rest of the math department, or mathematicians in general. (For example, in Brown's quote above, the listeners have been separated out as "you people" and are excluded from the *we*.)

Many of the examples of non-standard use of *we* fall into two patterns: *let's* , and *we + do* . The use of *let's* has an air of invitation about it, as well as the implication of a condescending baby talk register mentioned earlier.*We + do* usually seems to mean *I + do*. Tag questions, rhetorical questions, non-standard use of *we*, and "making encouraging noises" co-occur in many examples in the data, as they do in the baby talk register:

> Let's draw a picture over here. (Green)
>
> In order to see this -- initially we have to do it by brutal force. (Brown)
>
> Now let's see if we can see what's going on and if we can't, we're going to have to do more to it. (White)

There is a link between the non-standard use of *we* and the language of persuasion. Ferguson, the linguist who first defined the parameters of baby talk, writes, "The BT [baby talk] register often serves the purpose of coaxing or persuading...This 'wheedle' use ... can even be extended to objects or to unknown addressees", and cites " Alexander Woolcott's amusing habit of using BT in addressing his dice at backgammon. . . Another example of extension is its use in certain forms of advertising" (Ferguson, 1977, 231). Again, an analysis of discourse features of the introductory calculus lecture genre leads to its interpretation as a kind of persuasive talk.

4.4 The language of persuasion: new coinages and metaphors

Finally, I would like to consider is the role of metaphor and neologisms (new coinages) in calculus lectures. The numerous metaphors and neologisms in the data fell into two categories: personification and metaphors of physical movement over time:

> As soon as x gets a decent ways away from zero it's safe to divide by it. And if it's approaching minus infinity it won't be long before it leaves zero way behind. (White)
>
> Now a vertical asymptote is simply a vertical line like in this case the y-axis that the graph snuggles up to because it's having some kind of an infinite limit. And it's alright if it just snuggles up to it on one side, it's still an asymptote. And it doesn't have to snuggle up to both the positive and negative extensions of the line, just the one end of it. (White)

Metaphor is such an accepted feature of mathematical talk and math teaching that it would be hard to imagine a metaphor-less mathematics. It is clear that metaphor and neologisms play a role in describing, by association, new relationships and phenomena (and indeed it has been suggested that all natural languages are made up mostly of extended metaphors.) Having taken this into consideration, I suggest another use of metaphor in the context of the mathematics lecture - its use as a tool of persuasion.

Sandell, summarizing research into the persuasive effects of linguistic style, writes,

> Bowers and Osborn (1966) varied the final parts of two different speeches in two versions: one literal and one metaphorically intense. . . On both speeches, the effective difference between the versions was significant, the metaphorical ending increasing attitude change in the advocated direction. (Sandell, 1977, 77)

In their study of television advertising and televangelism as persuasive language, Schmidt and Kess identify neologisms as a key feature of persuasive discourse, since "persuasive discourse wears out; ordinary conversation does not" and because "anything neologistic… draws attention to itself, and by capturing the hearer's attention increases the impact of the message…Neologism forces the hearer to interpret, and therefore to participate in the discourse…This active role played by the hearer, in turn, enhances learning and retention, and consequently also persuasion" (Schmidt & Kess 1982, 30).

5. Conclusions

Linguistic analysis in this pilot study indicates that introductory university calculus lectures use the language of persuasion, the language of the hard sell to "sell" mathematics to an audience of students seen to be uninterested and uninvolved. An awareness of the persuasive "sales" nature of these lectures poses a number of problems for instructors in terms of pedagogy.

I argue that a sales pitch is an inappropriate model for educational talk. Students know that the "hard sell" in our culture represents a triumph of persuasion over criticism, reasoned thought and a person's "better judgement". The unstoppable stream of talk, the inexorable chain of logic forged in hard sell persuasion is well known to be a verbal cover-up for the deficiencies of a product or an opinion that would show its flaws in a more leisurely examination. The similarities of the introductory calculus lecture to a conjuring act also serve to raise doubts, since we know that the appearance of logic and inevitability in a magician's routine hides a series of tricks and deceptions. The discursive style of introductory calculus lectures may engage uninvolved students but simultaneously send the message that mathematics involves trickery and deception, and that one mustn't look at its mechanics too closely or ask too many questions. The tendency of such discourse to treat students as if they were unable or unwilling to respond is also inimical to critical or independent thought. The discourse form conveys an attitude of condescension and paternalism.

I suggest that university calculus lecturers need to pay attention to messages about mathematics learning that may be involuntarily conveyed by their style of discourse, and to work on developing new styles that both involve students and respect their ability to think and respond critically.

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MATHEMATICAL CONCEPTS, THEIR MEANINGS, AND UNDERSTANDING [1]

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> "Try not to think of understanding as a 'mental process' at all - For *that* is the expression which confuses you ( ) In the sense in which there are processess (including mental processes) which are characteristic of understanding, understanding is not a mental process" (Wittgenstein, 1953, Philosophical Investigations, p 61)

Abstract

Some key elements for developing a theory for understanding mathematical concepts are outlined. These elements are derived from the theory of mathematical objects and their meanings developed by Godino and Batanero (1994; 1996). We shall argue for the need to complement the psychological facets of understanding - 'as a mental experience', and 'connection of internal representations in information networks'- with the sociocultural approach, that is, understanding as 'correspondence between personal and institutional meanings'. The role of situation-problems and semiotic instruments is also emphasized in both personal and institutional dimensions of understanding processes.

Understanding in Mathematics Education

As Hiebert and Carpenter (1992) asserted, one of the most widely accepted ideas in mathematics education is that students should understand mathematics. Sierpinska (1994) starts her book on understanding in mathematics with similar words: "*how to teach so that students understand?. What exactly don't they understand? What do they understand and how?*" (p. XI). Pirie and Kieren (1994) mention the interest towards teaching and learning mathematics with understanding, which is shown in recent curricular reforms in many countries. This interest is also reflected in conference proceedings and research articles in psychology and artificial intelligence.

The importance of the idea of understanding for mathematics education is emphasized in recent works by Sierpinska, Pirie and Kieren, Koyama (1993), amongst others. Nevertheless, characterizing understanding "*in a way which highlights its growth, and identifying pedagogical acts which sponsor it, however, represent continuing problems*" (Pirie and Kieren, 1994, p 165).

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The use of the term 'understanding' (or 'comprehension') is varied, depending on institutional contexts, although the dominant psychological approach emphasizes the mental facet of understanding, which is strongly challenged by Wittgenstein. The cognitive revolution supported by Vygotsky -who claims the analytical and genetic priority of sociocultural factors when attempt to understand individual psychological processes-, Bruner (1990) -with his proposal of a cultural psychology, or Chevallard (1992) -who speaks of cognitive and didactical anthropology- requires a reconceptualization of mathematical knowledge and its understanding.

The book by Sierpinska (1994) represents an important step forward, when discerning between understanding acts and processes and when relating "*good understanding*" of a mathematical situation (concept, theory, problem) to the sequence of acts of overcoming obstacles specific to this situation. Through the historico-empirical approach it is possible to identify meaningful acts for understanding a concept. Nevertheless, we think that taking the notion of object as primitive and deriving meaning from understanding cause some difficulties in analyzing the processes of assessing students' understanding.

From our point of view, a theory of conceptual understanding useful for mathematics education should not be limited to saying, for example, that understanding the concept of function is a person's mental experience assigning some object to the term 'function'. A cultural entity, a very complex, not ostensive object is designated with the term 'function'. Therefore, in order to define what is understanding the concept of function, we need to clarify a previous question: What object must the student assign to the term 'function' so that the teacher may say that he/she understands the object function?

The problem of understanding is, consequently, closely linked to how the nature of mathematical knowledge is conceived. Mathematical terms and expressions denote abstract entities whose nature and origin should be researched for elaborating a useful and effective theory for what it is to understand such objects. This research requires answering questions such as: What is the structure of the object to be understood? What forms or ways of understanding exist for each concept? What are the possible and desirable aspects or components of mathematical concepts for students to learn at a given time and under certain circumstances? How are these components developed?

If, for example, we consider mathematical knowledge as internally represented information, understanding occurs when the representations achieved are connected by progressively more structured and cohesive networks (Hiebert and Carpenter, 1992). However, we consider it excessively reduccionist to view mathematical activity as information processing. From our point of view, the theories of understanding derived from this conception do not adequately describe the teaching and learning processes of mathematics, especially the social and cultural aspects involved in theses processes.

We think that a theory of understanding mathematical abstractions must be supported by a previous theory concerning the nature of such objects. In this research report, we first present a summary of our previous theoretical articles on the nature of the mathematical objects and its meanings (Godino and Batanero 1994; 1996). After this synthesis, we shall provide some elements deduced from this theory, to be taken into account for the development of a theory of understanding in the teaching and learning of mathematics.

Pragmatic and relativist ontosemantics for mathematics

Our theory is based on the following epistemological and cognitive assumptions about mathematics, which take into account some recent tendencies in the philosophy of mathematics (Tymoczko, 1986, Ernest, 1991):

a) Mathematics is a human activity involving the solution of problematic situations. In finding the responses or solutions to these external and internal problems, mathematical objects progressively emerge and evolve. According to Piagetian constructivist theories, people's acts must be considered the genetic source of mathematical conceptualization.

b) Mathematical problems and their solutions are shared in specific institutions or collectives involved in studying such problems. Thus, mathematical objects are socially shared cultural entities.

c) Mathematics is a symbolic language in which problem-situations and the solutions found are expressed. The systems of mathematical symbols have a communicative function and an instrumental role.

d) Mathematics is a logically organized conceptual system. Once a mathematical object has been accepted as a part of this system, it can also be considered as a textual reality and a component of the global structure. It may be handled as a whole to create new mathematical objects, widening the range of mathematical tools and, at the same time, introducing new restrictions in mathematical work and language.

To build our model, we take the notion of *problem-situation* as a primitive idea. Problems do not appear in isolation; the systematic variation of the variables intervening in problem-situations yields different *fields of problems*, sharing similar representations, solutions, etc.

The subject performs different types of *practices*, or actions intended to solve a mathematical problem, to communicate the solution to other people or to validate or generalize that solution to other settings and problems. The genesis of a subject's knowledge arises as a consequence of that subject's interaction with the field of problems, which is mediated by institutional contexts.

Two primary units of analysis to study cognitive and didactic processes are the *meaningful practices*, and the *meaning of an object*, for which we postulate two interdependent dimensions: personal and institutional. A practice is meaningful for a person (resp. institution) if it fulfills a function for solving the problem, or for communicating, validating, or extending the solution. This notion is used to conceptualize mathematical objects, both in their psychological and epistemological facets (personal and institutional objects). Mathematical objects -abstractions or empirico and operative generalizations (Dörfler, 1991)- are considered as emergents from the systems of personal (institutional) practices made by a person (or within an institution) when involved with some problem-situations.

The system of meaningful prototype practices, i.e., the system of efficient practices to reach the goal aimed at is defined as the *personal (institutional) meaning of the object*. It is considered to be the genetic (epistemological) origin of personal objects (institutional objects). It is linked to the field of problems from which this object emerges at a given time and it is a compound entity. Its nature is opposed to the intensional character of the object, and it allows us to focus, from another point of view, on the issues for designing teaching situations and assessing subjects' knowledge.

To sum up, we postulate a relativity of the emergent objects, intrinsic to the different institutions involved in the field of problems, and also depending on the available expressive forms. This assumption could be useful to explain the adaptations (or transpositions) and mutual influences that mathematical objects undergo when transmitted between people and institutions.

The systemic complexity we postulate for the meaning of mathematical concepts -understood from a pragmatic perspective - is well illustrated by the problem list that Sierpinska (1994, p. 21) outlines on the meaning of the term 'function', expressed by its use:

> *How* can a function be? Or, what adjectives can we use with the noun 'function' (Defined/ non-defined, defined in a point/ in an interval/ everywhere, increasing, decreasing, invertible, continuous in a point/ in an interval, smooth, differentiable in a point/ in a interval, integrable ..., etc). *What* can a function *have*? (Zeros, values, a derivative, a limit in a point/ in infinity, etc). *What* can be done, with functions? (Plot, calculate the values in points ..., calculate a derivative, an integral, combine functions, take sequences, series of functions, etc). How do we *verify* that a function is ... (continuous, differentiable ... in

> a point, increasing in an interval)? What can functions be *used for?* (Representing relations between variable magnitudes, modelling, predicting, interpolating, approximating,)

The socio-epistemic relativism for the meaning of mathematical abstractions that we postulate in our theoretical model is justified, for example, when we study the diversity of *conceptions* of the function concept identified by Ruiz-Higueras (1994), not only from a historical perspective, but also in didactic transpositions at the teaching institutions (curricula, text books, mathematics classroom).

Elements for a model of understanding in Mathematics Education

From our theoretical positions summarized in the previous Section, we identify the following consequences that should be considered in order to elaborate a theory on understanding mathematics.

Institutional and personal dimension

According to our pragmatic and relativist conception of mathematics, a theory of mathematical understanding, which is to be useful and effective to explain teaching and learning processes, should recognize the dialectical duality between the personal and institutional facets of knowledge and its understanding.

The definition of understanding by Sierpinska as the 'mental experience of a subject by which he/she relates an object (sign) to another object (meaning)' emphasizes one of the senses in which the term 'understanding' is used, well adapted for studying the psychological processes involved. Nevertheless, in mathematics teaching the term 'understanding' is also used in the processes for assessing students' learning. School institutions expect subjects to appropiate some culturally fixed objects, and assign the teacher with the task of helping the students to establish the agreed relationships between terms, mathematics expressions, abstractions, and techniques. In this case, understanding is not merely a mental activity, but it is converted into a social process. As an example, we may consider that a pupil sufficiently "understands" the function concept in secondary teaching and that he/she does not understand it, if the judgment is made by a university institution.

Furthermore, from a subjective sense, understanding cannot merely be reduced to a mental experience but it involves the person's whole world. As Johnson (1987) states, our understanding *"is the way we are meaningfully situated in our world through our bodily interactions, our cultural institutions, our linguistic tradition, and our historial context"* (p. 102).

Systemic nature

Since, in our theoretical model, we start out from the notions of object and meaning, personal understanding of a concept is "grasping or acquiring the meaning of the object". Therefore, the construct 'meaning of an object' is not conceived as an absolute and unitary entity, but rather as compound and relative to institutional settings. Therefore, the subject's understanding of a concept, at a given moment and under certain circumstances, will imply the appropriation of the different elements composing the corresponding institutional meanings:

- extensional elements (recognition of prototype situations of use of the object);
- intensional elements (different characteristic properties and relationships with other entities);
- expressions and symbolic notations used to represent situations, properties, and relationships.

Furthermore, recognizing the systemic complexity of the object's meaning implies a dynamical, progressive, though nonlinear nature of the process of appropriation by the subject (Pirie & Kieren, 1994), due to the different domains of experience and institutional contexts in which he/she participates.

The conception of mathematics underlying our theoretical model is characterized primarily by considering mathematics as a human activity. Concepts and mathematical procedures emerge from a persons' acts for solving some problem fields. This activity is mediated by the semiotic instruments provided by the culture and by our capacity for deductive logical reasoning. Secondly, mathematics is a socially shared and logically structured conceptual system. Consequently, the process axis for personal understanding must contain the following categories: intuitive (operative), declaratory (communicative), argumentative (validating), and structural (institutionalized). The achievement of these levels of understanding for a concept or conceptual field will require the organization of specific didactic moments or situations, as Brousseau (1986) proposes in his *didactic situations theory*.

Human action and intentionality

Our theoretical model also includes, as the primary unit of analysis, the notion of *meaningful prototype practice*, defined as the action that the person carries out in his/her attempts for solving a class of problem-situations and for which he/she recognizes or attributes a purpose (an *intentionality*). Therefore, this is a situated expressive form involving a problem-situation, an institutional context, a person and the semiotic instruments mediating the action. This notion is used to define mathematical objects as emerging from the systems of meaningful prototype practices. Consequently, understanding the object, in its integral or systemic sense, requires the subject, not only the semiotic and relational components, but to identify a *role* -an *intention* (Maier, 1988)- in the problem solving process for the object.

Assessment of understanding

We conceive the processes for assessing understanding as the study of the correspondence between personal and institutional meanings. The evaluation of a subject's understanding is relative to the institutional contexts in which the subject participates. An institution (educational or not) will say that a subject "understands" the meaning of an object - or that he/she 'has grasped the meaning' of a concept, if the subject is able to carry out the different prototype practices that make up the meaning of the institutional object.

It is also necessary to recognize the *unobservable construct character* of personal understanding. Consequently, an individual's personal understanding about a mathematical object may be deduced from the analysis of the practices carried out by the person in solving problematic tasks, which are characteristics of that object. Since, for each mathematical object, the population of such tasks is potentially unlimited, the analysis of the task variables and the selection of the items to design evaluation instruments become of primary interest. The construct 'meaning of an object' we propose, in its two dimensions, personal and institutional, might be a useful conceptual tool to study the evaluation processes, the achievement of the 'good understanding', and the institutional and evolutionary factors conditioning them.

Final remarks

In this report, we explore some elements for a theory of understanding in mathematics, derived from our previous research work into the meaning of mathematical objects.

Our proposal is that mental functioning and sociocultural settings be understood as dialectically interacting moments, or aspects of a more inclusive unit of analysis: *human action*. Nevertheless, it is necessary to recognize the analitical priority of studying institutional meanings. Since each person developes in different institutions and cultural settings, the psychological processes involved in understanding the linguistic or conceptual mathematical objects are mediated by institutional meanings, namely, by situations-problems, semiotic instruments, habits and shared conventions.

Finally, the systemic nature of meaning and understanding highlights the sampling character of teaching and assessment situations, and the inferential problems associated with their study. Therefore, we propose the characterization of the personal and institutional meaning of mathematical objects, and of its mutual interdependence and development as a priority research agenda for Mathematics Education (Godino and Batanero, 1996).

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